DYNAMICAL PROPERTIES OF THE DERIVATIVE OF THE WEIERSTRASS ELLIPTIC FUNCTION

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ABSTRACT. We discuss properties of the Julia and Fatou sets of the derivative of the Weierstrass elliptic $\wp$ function. We find triangular lattices for which the Julia set is the whole sphere, or which have superattracting fixed or period two points. We study the parameter space of the derivative of the Weierstrass elliptic function on triangular lattices and explain the symmetries of that space.

1. INTRODUCTION

The study of complex dynamical systems began in the early 1900s with the work of mathematicians such as Fatou [10] and Julia [17]. These works focused on the iteration of rational functions, although Fatou later published articles on the iteration of entire functions [11]. In 1988, Devaney and Keen [5] published the first paper investigating the dynamics of a transcendental meromorphic function. Since then, it has been well established that transcendental meromorphic functions can exhibit dynamical behavior distinct from that of rational maps ([1] - [6], [9] ,[12] – [16], [21] – [23]).

Studies on the dynamical, measure-theoretic, and topological properties of iterated elliptic functions have appeared in [12] – [16], [21] – [23]. The Weierstrass elliptic $\wp$-function satisfies some strong algebraic identities which influence the resulting dynamical behavior. Even within an equivalence class of lattice shape, changing the size or orientation of the lattice can drastically change the dynamics of the Weierstrass elliptic function. Most of the work investigating the dynamics of parametrized families of elliptic functions involve the study of the Weierstrass elliptic function $\wp$ ([12] – [16]).

In this paper, we investigate the dynamics of the derivative of the Weierstrass elliptic function, focusing mainly on triangular lattices. Although the Weierstrass elliptic function and its derivative share some of the same algebraic properties, moving from the order two elliptic

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function $\wp$ to the order three function $\wp'$ changes many of the dynamical properties. For example, on a triangular lattice Ω, $\wp_\Omega$ has three distinct critical values, but $\wp'_\Omega$ only has two. As the postcritical orbits strongly influence the dynamical properties of an iterated family, $\wp_\Omega$ and $\wp'_\Omega$ exhibit different types of behavior.

Here, we construct lattices for which the Julia set of $\wp'_\Omega$ is the entire sphere, and we also construct lattices for which $\wp'_\Omega$ has a superattracting fixed point or a superattracting period two cycle. We also investigate the symmetries of the parameter space arising from all triangular lattices.

The paper is organized as follows. In Sections 2 and 3 we give background on the dynamics of meromorphic functions and lattices in the plane. In Section 4, we define the function that we study, the derivative of the Weierstrass elliptic function, and discuss the location of the critical points and critical values of this function. In Section 5 we discuss the symmetries of the Fatou and Julia sets that arise from the algebraic properties of these elliptic functions. Section 6 focuses on the postcritical set of $\wp'_\Omega$ when Ω is a triangular lattice. In this section, we construct many triangular lattices Ω for which the postcritical set $\wp'_\Omega$ exhibits especially nice behavior.

In Section 7 we discuss parametrizing the derivative of the Weierstrass elliptic function over all triangular lattices. We find a subset of this parameter space which gives a reduced holomorphic family, and we discuss symmetries of parameter space that arise from other dynamical properties of this family of maps.

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2. Background on the Dynamics of Meromorphic Functions

Let $f\colon \mathbb{C} \to \mathbb{C}_\infty$ be a meromorphic function where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. The *Fatou set* $F(f)$ is the set of points $z \in \mathbb{C}_\infty$ such that $\{f^n \colon n \in \mathbb{N}\}$ is defined and normal in some neighborhood of z. The *Julia set* is the complement of the Fatou set on the sphere, $J(f) = \mathbb{C}_\infty \backslash F(f)$. Notice that $\mathbb{C}_\infty \backslash \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}$ is the largest open set where all iterates are defined. If f has at least one pole that is not an omitted value, then $\overline{\bigcup_{n\geq 0} f^{-n}(\infty)}$ has more than two elements. Since $f(\mathbb{C}_\infty \backslash \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}) \subset \mathbb{C}_\infty \backslash \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}$, Montel's theorem implies that

$$J(f) = \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}.$$

Let $Crit(f)$ denote the set of critical points of f, *i.e.*,

$$Crit(f) = \{z\colon f'(z) = 0\}.$$

If z_0 is a critical point then $f(z_0)$ is a *critical value.* The *postcritical set* of f is:

$$P(f) = \overline{\bigcup_{n\geq 0} f^{n}(Crit(f))}.$$

A point z_0 is *periodic* of period p if there exists a $p \geq 1$ such that $f^p(z_0) = z_0$. We also call the set $\{z_0, f(z_0), \ldots, f^{p-1}(z_0)\}$ a *p-cycle.* The *multiplier* of a point z_0 of period p is the derivative $(f^p)'(z_0)$. A periodic point z_0 is classified as *attracting, repelling,* or *neutral* if $|(f^p)'(z_0)|$ is less than, greater than, or equal to 1 respectively. If $|(f^p)'(z_0)| = 0$ then z_0 is called a *superattracting* periodic point. As in the case of rational maps, the Julia set is the closure of the repelling periodic points [1].

Suppose U is a connected component of the Fatou set. We say that U is *preperiodic* if there exists $n > m \geq 0$ such that $f^n(U) = f^m(U)$, and the minimum of $n - m = p$ for all such n, m is the *period* of the cycle. Although elliptic functions with a finite number of critical values are meromorphic, it turns out that the classification of periodic components of the Fatou set is no more complicated than that of rational maps of the sphere. Periodic components of the Fatou set of these elliptic functions may be attracting domains, parabolic domains, Siegel disks, or Herman rings ([3], [9], [13]).

Let $C = \{U_0, U_1, \ldots U_{p-1}\}$ be a periodic cycle of components of $F(f)$. If C is a cycle of immediate attractive basins or parabolic domains, then $U_j \cap Crit(f) \neq \emptyset$ for some $0 \leq j \leq p-1$. If C is a cycle of Siegel Disks or Herman rings, then $\partial U_j \subset \overline{\bigcup_{n\geq 0} f^n(Crit(f))}$ for all $0 \leq j \leq p-1$. In particular, critical points are required for any type of preperiodic Fatou component.

3. Background on Lattices in the Plane

Let $\omega_1,\, \omega_2 \in \mathbb{C}\backslash\{0\}$ such that $\omega_2/\omega_1 \notin \mathbb{R}$. We define a lattice of points in the complex plane by $\Omega = [\omega_1, \omega_2]\colon = \{m\omega_1 + n\omega_2\colon m, n \in \mathbb{Z}\}$. It is well-known that two different sets of vectors can generate the same lattice Ω. If $\Omega = [\omega_1, \omega_2]$, then any other generators η_1, η_2 of Ω are obtained by multiplying the vector (ω_1, ω_2) by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. The values $\omega_3 = \omega_1 + \omega_2$ and $\omega_4 = \frac{1}{2}\omega_3$ will be used later in this paper.

We can view Ω as a group acting on $\mathbb{C}$ by translation, each $\omega \in \Omega$ inducing the transformation of $\mathbb{C}$:

$$T_\omega : z \mapsto z + \omega.$$

Definition 3.1. *A closed, connected subset Q of $\mathbb{C}$ is defined to be a* fundamental region *for Ω if*

(1) *for each $z \in \mathbb{C}$, Q contains at least one point in the same Ω-orbit as z;*
(2) *no two points in the interior of Q are in the same Ω-orbit.*

If Q is any fundamental region for Ω, then for any $s \in \mathbb{C}$, the set

$$Q + s = \{z + s : z \in Q\}$$

is also a fundamental region. Usually (but not always) we choose Q to be a polygon with a finite number of parallel sides, in which case we call Q a period parallelogram *for Ω.*

Frequently we refer to types of lattices by the shapes of the corresponding period parallelograms. If Ω is a lattice, and $k \neq 0$ is any complex number, then $k\Omega$ is also a lattice defined by taking $k\omega$ for each $\omega \in \Omega$; $k\Omega$ is said to be *similar* to Ω. Similarity is an equivalence relation between lattices, and an equivalence class of lattices is called a *shape*.

Let $\overline{\Omega}$ denote the set of complex numbers $\overline{\omega}$ for all $\omega \in \Omega$. Then $\overline{\Omega}$ is also a lattice. If $\Omega = \overline{\Omega}$, Ω is called a *real lattice*. There are two special lattice shapes: square and triangular. A *square* lattice is a lattice with the property that $i\Omega = \Omega$. A *triangular* lattice is a lattice with the property that $\varepsilon\Omega = \Omega$, where ε is a cube root of unity; such a lattice forms a pattern of equilateral triangles throughout the plane. A triangular lattice is in the horizontal position if the main axis of the rhombus is parallel to the real axis, and vertical if the main axis is parallel to the imaginary axis.

Among all lattices, those having the most regular period parallelograms are distinguished in many respects. For example, results on how the lattice shape influences the dynamics of the Weierstrass elliptic function can be found in [12] – [16].

4. The Derivative of the Weierstrass Elliptic Function

For any $z \in \mathbb{C}$ and any lattice Ω, the Weierstrass elliptic function is defined by

$$\wp_\Omega(z) = \frac{1}{z^2} + \sum_{\omega \in \Omega \backslash \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

Replacing every z by $-z$ in the definition we see that $\wp_\Omega$ is an even function. It is well-known that $\wp_\Omega$ is meromorphic and is periodic with respect to Ω.

The derivative of the Weierstrass elliptic function is given by

$$\wp'_\Omega(z) = -2 \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^3}.$$

It is also an elliptic function and is periodic with respect to Ω. It is clear from the series definition that $\wp'_\Omega$ is an odd function. In addition, $\wp'_\Omega$ is also meromorphic, with poles of order three at lattice points.

The Weierstrass elliptic function and its derivative are related by the differential equation

$$(\wp'_\Omega(z))^2 = 4(\wp_\Omega(z))^3 - g_2 \wp_\Omega(z) - g_3, \tag{1}$$

where $g_2(\Omega) = 60 \sum_{\omega \in \Omega \backslash \{0\}} \omega^{-4}$ and $g_3(\Omega) = 140 \sum_{\omega \in \Omega \backslash \{0\}} \omega^{-6}$. The numbers $g_2(\Lambda)$ and $g_3(\Lambda)$ are invariants of the lattice Λ in the following sense: if $g_2(\Lambda) = g_2(\Lambda')$ and $g_3(\Lambda) = g_3(\Lambda')$, then $\Lambda = \Lambda'$. Furthermore, given any g_2 and g_3 such that $g_2^3 - 27g_3^2 \neq 0$ there exists a lattice Λ having $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ as its invariants. If Λ is a square lattice then $g_2 = 0$, and if Λ is a triangular lattice, $g_3 = 0$ [8].

It will be useful to have an expression for $\wp''_\Omega$, the second derivative of the Weierstrass elliptic function for a given lattice Ω. Starting with $\wp'_\Omega(z) = -2 \sum_\Omega (z-\omega)^{-3}$, we differentiate term by term to find that

$$\wp''_\Omega(z) = 6 \sum_\Omega (z-\omega)^{-4}.$$

Further, using Equation 1, we have that

$$\wp''_\Omega(z) = 6(\wp_\Omega(z))^2 - \frac{g_2(\Omega)}{2}. \tag{2}$$

The Weierstrass elliptic function, its derivatives, and the lattice invariants satisfy the following homogeneity properties.

Proposition 4.1. *For any lattice Ω and for any $m \in \mathbb{C} \backslash \{0\}$,*

$$\wp_{m\Omega}(mz) = m^{-2} \wp_\Omega(z),$$

$$\wp'_{m\Omega}(mz) = m^{-3} \wp'_\Omega(z),$$

$$\begin{aligned} \wp''_{m\Omega}(mz) &= m^{-4}\wp''_{\Omega}(z), \\ g_2(m\Omega) &= m^{-4}g_2(\Omega), \\ g_3(m\Omega) &= m^{-6}g_3(\Omega). \end{aligned}$$

Verification of the homogeneity properties can be seen by substitution into the series definitions.

The homogeneity property of $\wp'_{\Omega}$ influences the behavior of $\wp'_{\Omega}$ under iteration.

Corollary 4.2. *If Ω is any lattice then*

$$(\wp'_{\Omega})^n(-z) = -(\wp'_{\Omega})^n(z).$$

Proof. Since $\Omega = -\Omega$ for any lattice, the result follows from the homogeneity property in Proposition 4.1. □

Critical points and values play an important role in complex dynamics, so it is useful for us to be able to locate these points for $\wp_{\Omega}$ and $\wp'_{\Omega}$. From [8], we have that the critical points of $\wp_{\Omega}$ lie exactly on the half lattice points of Ω; that is, on $\omega_j/2 + \Omega$ for $j \in \{1, 2, 3\}$. We discuss the critical points for $\wp'_{\Omega}$ in the following proposition.

Proposition 4.3. [8] *The critical points of $\wp'$ are the points where $\wp^2(z) = \dfrac{g_2}{12}$. Further, the critical values of $\wp'$ are $\pm\{-g_3 \pm (\dfrac{g_2}{3})^{3/2}\}^{1/2}$.*

Proof. We have $\wp''(z) = 6(\wp(z))^2 - g_2/2$ from Equation 2. Solving $\wp''(z) = 0$ gives us that $\wp'$ has critical points in the four congruence classes where $(\wp(z))^2 = g_2/12$.

The critical values of $\wp'$ are found by solving $4(\wp(z))^3 - g_2\wp(z) - ((\wp'(z))^2 + g_3) = 0$ for $\wp'(z)$ after substituting $\pm\sqrt{g_2/12}$ for $\wp(z)$. Thus

$$4\sqrt{\frac{g_2}{12}}^3 - g_2\sqrt{\frac{g_2}{12}} \pm ((\wp'(z))^2 + g_3) = 0,$$

which implies that

$$-\frac{1}{3}g_2^{3/2} = \pm\sqrt{3}((\wp'(z))^2 + g_3).$$

Squaring both sides and rearranging terms shows that ${g_2}^3 - 27((\wp'(z))^2 + g_3)^2 = 0$, and by solving for $\wp'(z)$, we see that the critical values of $\wp'(z)$ are $\pm\{-g_3 \pm (\dfrac{g_2}{3})^{3/2}\}^{1/2}$. □

The critical values are distinct unless the lattice is triangular.

Corollary 4.4. *If Ω is triangular then Ω has exactly two equivalence classes of critical points at $\pm\dfrac{1}{3}\omega_3 + \Omega = \pm\dfrac{1}{3}(\omega_1 + \omega_2) + \Omega = \pm\dfrac{2}{3}\omega_4 + \Omega$ and two distinct critical values at $\pm\sqrt{-g_3}$.*

Proof. We note that if Ω is triangular then the critical points coincide in pairs with the zeros of $\wp_\Omega(z)$. These points occur at the points at the center of the equilateral triangles determined by the lattice, $\pm\frac{1}{3}\omega_3 + \Omega = \pm\frac{2}{3}\omega_4 + \Omega$. Since $g_2(\Omega) = 0$ for triangular lattices there are only two critical values by Proposition 4.3. $\square$

5. Properties of the Julia and Fatou Sets of $\wp'_\Omega$

We begin our investigation of the dynamics of $\wp'_\Omega$ for an arbitrary lattice Ω with an examination of the symmetries that arise in the Julia and Fatou sets.

Theorem 5.1. *If Ω is any lattice then*

1. $F(\wp'_\Omega) = F(\wp'_\Omega) + \Omega$ *and* $J(\wp'_\Omega) = J(\wp'_\Omega) + \Omega$.
2. $F(\wp'_\Omega) = -1F(\wp'_\Omega)$ *and* $J(\wp'_\Omega) = -1J(\wp'_\Omega)$.
3. $F(\wp'_{\overline{\Omega}}) = \overline{F(\wp'_\Omega)}$ *and* $J(\wp'_{\overline{\Omega}}) = \overline{J(\wp'_\Omega)}$.
4. *If Ω is square, then* $F(\wp'_\Omega) = iF(\wp'_\Omega)$ *and* $J(\wp'_\Omega) = iJ(\wp'_\Omega)$.
5. *If Ω is triangular, then* $\varepsilon F(\wp'_\Omega) = F(\wp'_\Omega)$ *and* $\varepsilon J(\wp'_\Omega) = J(\wp'_\Omega)$ *where ε is a cube root of unity.*

Proof. The proof of (1) follows immediately from the periodicity of $\wp'_\Omega$ with respect to Ω.

For (2), let $z \in F(\wp'_\Omega)$. By definition, $(\wp'_\Omega)^n(z)$ exists and is normal for all n. Let U be a neighborhood of z such that $\{(\wp'_\Omega)^n(U)\}$ forms a normal family. Let $V = -U$. By Corollary 4.2, we have that $(\wp'_\Omega)^n(V) = -(\wp'_\Omega)^n(U)$ for all $n \geq 1$ and thus $\{(\wp'_\Omega)^n(V)\}$ forms a normal family. The proof of the converse is identical. So $z \in F(\wp'_\Omega)$ if and only if $-z \in F(\wp'_\Omega)$, and the Fatou set is symmetric with respect to the origin. This of course forces the Julia set to be symmetric with respect to the origin as well.

To prove (3), define $\phi(z) = \overline{z}$. We see that $\phi \circ \wp'_\Omega = \wp'_{\overline{\Omega}} \circ \phi$ for all lattices Ω, so for a general lattice the map $\wp'_\Omega$ is conjugate to $\wp'_{\overline{\Omega}}$, and the Julia sets are conjugate under ϕ.

For (4), we know that square lattices satisfy $i\Omega = \Omega$. Using Proposition 4.1, we have

$$(\wp'_\Omega)^n(-iz) = (\wp'_{i\Omega})^n(-iz) = -i(\wp'_\Omega)^n(z).$$

for all $n \geq 0$. Thus the Julia set and the Fatou set of a square lattice must be symmetric with respect to rotation by $\pi/2$.

A similar application of the homogeneity lemma proves (5). If ε is a cube root of one, then so is $\varepsilon^2 = 1/\varepsilon$; thus $\varepsilon^2\Omega = \Omega$. Then from Proposition 4.1, $\wp'_\Omega(\varepsilon^2 z) = \wp'_{\varepsilon^2\Omega}(\varepsilon^2 z) = \wp'_\Omega(z)$; by induction, $(\wp'_\Omega)^n(\varepsilon^2 z) = (\wp'_\Omega)^n(z)$. $\square$

In addition to a basic Julia set pattern repeating on each fundamental region, we also see symmetry within the period parallelogram.

Proposition 5.2. *For the lattice $\Omega = [\omega_1, \omega_2]$, $J(\wp'_\Omega)$ and $F(\wp'_\Omega)$ are symmetric with respect to the half lattice points $\omega_1/2 + \Omega$, $\omega_2/2 + \Omega$, and $(\omega_1 + \omega_2)/2 + \Omega$.*

Proof. This follows easily from Theorem 5.1 (1), (2). We have that $z \in J(\wp'_\Omega)$ if and only if $-z + \Omega \in J(\wp'_\Omega)$, and a half lattice point must lie between z and $-z + \Omega$ for any element of the lattice. □

6. Postcritical Orbits

Recall from Proposition 4.3 that the critical points of $\wp'$ are the points where $\wp^2(u) = g_2/12$. Our next result shows that multiplying the lattice Ω by k changes the location of the critical points from a_Ω to ka_Ω.

Theorem 6.1. *Let Ω be a lattice and suppose a_Ω is a critical point of $\wp'_\Omega$. Then ka_Ω is a critical point of $\wp'_{k\Omega}$.*

Proof. Suppose a_Ω is a critical point for $\wp'_\Omega$; that is, assume that $[\wp_\Omega(a_\Omega)]^2 = g_2(\Omega)/12$. From Proposition 4.1, we have that $g_2(k\Omega) = k^{-4}g_2(\Omega)$ and

$$[\wp_{k\Omega}(ka_\Omega)]^2 = [\frac{1}{k^2}\wp_\Omega(a_\Omega)]^2 = \frac{1}{k^4}[\wp_\Omega(a_\Omega)]^2 = \frac{g_2(\Omega)}{k^4 12} = \frac{g_2(k\Omega)}{12}.$$

Thus if a_Ω is a critical point of $\wp'_\Omega$, then ka_Ω is a critical point of $\wp'_{k\Omega}$. □

We will use the notation $a_{k\Omega}$ to denote the critical point ka_Ω for $\wp'_{k\Omega}$.

From Corollary 4.4, we know that triangular lattices are distinguished by the fact that they have exactly two critical values. We restrict our attention to triangular lattices throughout the rest of the paper. In particular, the postcritical orbits of $\wp'_\Omega$ are related in an especially nice way when Ω is a triangular lattice.

Proposition 6.2. *If Ω is a triangular lattice then $P(\wp'_\Omega)$ is contained in two forward invariant sets: one set*

$$\alpha = \overline{\bigcup_{n \geq 0} (\wp'_\Omega)^n(a_\Omega)},$$

and the set $e^{i\pi}\alpha$. (These sets are not necessarily disjoint.)

Proof. The proof follows from the application of Corollaries 4.2 and 4.4. □

Let Λ be the lattice generated by $g_2 = 0$, $g_3 = -4$. We call Λ the *standard triangular lattice*, and we reserve the symbol Λ to denote this particular lattice throughout the rest of this paper. Then Λ is a triangular lattice in the horizontal position. Let λ_1, λ_2 be a pair of generators for this lattice such that λ_1 is in the first quadrant and λ_2 is its conjugate in the fourth quadrant. Using Mathematica [25] or the tables in [27], we can estimate $\lambda_1 \approx 2.1+1.2i$ and $\lambda_2 \approx 2.1-1.2i$. Define $\lambda_3 = \lambda_1 + \lambda_2$ and $\lambda_4 = \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{2}\lambda_3$. Note that both λ_3 and λ_4 are real. Recall from Corollary 4.4 that for a general triangular lattice Ω, the critical points are $\pm\frac{2}{3}\omega_4$; let $a_\Lambda = \frac{2}{3}\lambda_4$ so that a_Λ is a critical point of $\wp'_\Lambda$. For the lattice $k\Lambda$, let $k\lambda_n$ denote the lattice points for $n = 1, 2, 3$, and let $a_{k\Lambda}$ denote the critical point ka_Λ. Theorem 6.1 gives that $a_{k\Lambda} = \frac{2k}{3}\lambda_4$. Since any triangular lattice Ω can be written as $\Omega = k\Lambda$ for some k, our discussion will now focus on the lattice Λ.

We begin with a lemma explaining how multiplying the standard triangular lattice Λ by certain values of k changes the critical values of $\wp'_{k\Lambda}$. The lemma will be useful in finding lattices for which the postcritical orbit is especially simple.

Lemma 6.3. *Let Λ be the standard triangular lattice, j be a nonzero integer, and choose k such that $k^4 = (\frac{2}{j})\frac{-2}{\lambda_3}$. Then $\wp'_{k\Lambda}(ka_\Lambda) = \frac{j}{2}k\lambda_3$. That is, $\wp'_{k\Lambda}$ maps the critical point $a_{k\Lambda}$ to the critical value $(\frac{j}{2})k\lambda_3$.*

Proof. Note that $\wp'_\Lambda(a_\Lambda) = -\sqrt{-g_3(\Lambda)} = -2$. By Proposition 4.1, $\wp'_{k\Lambda}(ka_\Lambda) = \frac{1}{k^3}\wp'_\Lambda(a_\Lambda)$. Multiplying by $\frac{k}{k}$ gives

$$\frac{k}{k^4}\wp'_\Lambda(a_\Lambda) = \frac{-2k}{(\frac{2}{j})\frac{-2}{\lambda_3}} = (\frac{j}{2})k\lambda_3 = \frac{j}{2}k\lambda_3$$

as desired. □

We can use Lemma 6.3 to find values of k so that $\wp'_{k\Lambda}$ has critical values located at either lattice points or half lattice points.

Lemma 6.4. *Let Λ be the standard triangular lattice, and let j be an even, nonzero integer. Choose k such that $k^4 = (\frac{2}{j})\frac{-2}{\lambda_3}$. Then $\wp'_{k\Lambda}$ maps the critical point $a_{k\Lambda}$ to a lattice point of $k\Lambda$.*

Proof. We know from Lemma 6.3 that $\wp'_{k\Lambda}(ka_\Lambda) = \frac{j}{2}k\lambda_3$. Because j is even, $\frac{j}{2}k\lambda_3$ is a lattice point of $k\Lambda$. □

Lemma 6.5. *Let Λ be the standard triangular lattice, and let j be an odd integer. Choose k such that $k^4 = (\frac{2}{j})\frac{-2}{\lambda_3}$. Then $\wp'_{k\Lambda}$ maps the critical point $a_{k\Lambda}$ to a half lattice point of $k\Lambda$.*

Proof. We know from Lemma 6.3 that $\wp'_{k\Lambda}(ka_\Lambda) = \frac{j}{2}k\lambda_3$. Because j is odd, $\frac{j}{2}k\lambda_3$ is a half lattice point of $k\Lambda$. □

We can use the previous lemmas to find lattices for which all of the critical points are prepoles and thus lie in the Julia set.

Theorem 6.6. *Let Λ be the standard triangular lattice, and choose k so that $k^4 = (\frac{2}{j})\frac{-2}{\lambda_3}$ for some nonzero integer j. Then the Julia set of $\wp'_{k\Lambda}$ is $\mathbb{C}_\infty$.*

Proof. Suppose j is odd. By Corollary 6.5, $\wp'_{k\Lambda}(ka_\Lambda)$ lands on a half lattice point of $k\Lambda$. Recall that the critical points of $\wp$ lie at half lattice points; thus $\wp'_{k\Lambda}(\wp'_{k\Lambda}(ka_\Lambda)) = 0$. Then $a_{k\Lambda}$ is a pre-pole, and by Corollary 4.2 so is $-a_{k\Lambda}$. Then the postcritical set $\{0, \infty\}$ is a finite subset of $J(\wp'_{k\Lambda})$, and thus $J(\wp'_{k\Lambda}) = \mathbb{C}_\infty$.

Now suppose j is even. By Lemma 6.4, $\wp'_{k\Lambda}(ka_\Lambda)$ lands on a lattice point of $k\Lambda$. But the lattice points of $k\Lambda$ are the poles of $\wp'_{k\Lambda}$, so $\wp'_{k\Lambda}(ka_\Lambda)$ is a pole. Again we have a finite postcritical set contained in $J(\wp'_{k\Lambda})$, and thus $J(\wp'_{k\Lambda}) = \mathbb{C}_\infty$. □

Next, we focus on finding specific values of k which will map critical points to critical points, which we can use to find examples where the Julia set is not the entire sphere. We begin with a lemma that describes how to map critical points to integer multiples of critical points.

Lemma 6.7. *Let Λ be the standard triangular lattice and m be a nonzero integer. Choose k such that $k^4 = \frac{-2}{ma_\Lambda}$. Then $\wp'_{k\Lambda}(a_{k\Lambda}) = ma_{k\Lambda}$.*

Proof. By Proposition 4.1, $\wp'_{k\Lambda}(ka_\Lambda) = \frac{1}{k^3}\wp'_\Lambda(a_\Lambda)$. Again, we multiply by $\frac{k}{k}$ and have

$$\frac{k}{k^4}\wp'_\Lambda(a_\Lambda) = \frac{-2k}{\frac{-2}{ma_\Lambda}} = mka_\Lambda = ma_{k\Lambda}$$

as desired. □

The role of the integer m is similar to that of the integer j in Lemma 6.3: different values of m give rise to different consequences.

Lemma 6.8. *Let Λ be the standard triangular lattice, and let m be a nonzero integer of the form $3n$. Choose k such that $k^4 = \frac{-2}{ma_\Lambda}$. Then $\wp'_{k\Lambda}(ka_\Lambda)$ lands on a lattice point of $k\Lambda$.*

Proof. From Lemma 6.7, $\wp'_{k\Lambda}(ka_\Lambda) = ma_{k\Lambda}$. Then $ma_{k\Lambda} = 3n\frac{2}{3}\lambda_4 = n\lambda_3$. □

Note that this is the same case as in Lemma 6.4.

Next, we show that if m has the form $3n+1$ then $\wp'_{k\Lambda}$ has two superattracting fixed points.

Lemma 6.9. *Let Λ be the standard triangular lattice, and let m be a nonzero integer of the form $3n+1$. Choose k such that $k^4 = \frac{-2}{ma_\Lambda}$. Then $ma_{k\Lambda}$ and $-ma_{k\Lambda}$ are superattracting fixed points for $\wp'_{k\Lambda}$.*

Proof. We know from Lemma 6.7 that $\wp'_{k\Lambda}(ka_\Lambda) = ma_{k\Lambda}$. Because m has the form $3n+1$, $ma_{k\Lambda} = (3n+1)a_{k\Lambda} = 3n\frac{1}{3}k\lambda_3 + \frac{2}{3}k\lambda_4 = nk\lambda_3 + \frac{2}{3}k\lambda_4 \cong \frac{2}{3}k\lambda_4 = a_{k\Lambda}$. Hence we see that $ma_{k\Lambda}$ and $a_{k\Lambda}$ are in the same residue class and thus map to the same point. Thus $\wp'_{k\Lambda}(ma_{k\Lambda}) = \wp'_{k\Lambda}(a_{k\Lambda}) = ma_{k\Lambda}$, and we see that $ma_{k\Lambda}$ is a super attracting fixed point of $\wp'_{k\Lambda}$. Since $\wp'_{k\Lambda}$ is odd, $-ma_{k\Lambda}$ is also a super attracting fixed point of $\wp'_{k\Lambda}$ □

On the other hand, if m has the form $3n-1$ then $\wp'_{k\Lambda}$ has a superattracting two-cycle.

Lemma 6.10. *Let Λ be the standard triangular lattice, and let m be a nonzero integer of the form $3n-1$. Choose k such that $k^4 = \frac{-2}{ma_\Lambda}$. Then $\{ma_{k\Lambda}, -ma_{k\Lambda}\}$ form a superattracting 2-cycle for $\wp'_{k\Lambda}$.*

Proof. We know from Lemma 6.7 that $\wp'_{k\Lambda}(ka_\Lambda) = ma_{k\Lambda}$. Since m has the form $3n-1$, $ma_{k\Lambda} = (3n-1)a_{k\Lambda} = 3n\frac{1}{3}k\lambda_3 - \frac{2}{3}k\lambda_4 = nk\lambda_3 - \frac{2}{3}k\lambda_4 \cong -\frac{2}{3}k\lambda_4 = -a_{k\Lambda}$. Thus we have that $ma_{k\Lambda}$ and $-a_{k\Lambda}$ are congruent (mod $k\Lambda$). Then, by Proposition 4.1, we see that $\wp'_{k\Lambda}(ma_{k\Lambda}) = \wp'_{k\Lambda}(-a_{k\Lambda}) = -\wp'_{k\Lambda}(a_{k\Lambda}) = -ma_{k\Lambda}$. Similarly, $\wp'_{k\Lambda}(-ma_{k\Lambda}) = ma_{k\Lambda}$, and we have a superattracting 2-cycle. □

The next theorem follows immediately from the previous two lemmas.

Theorem 6.11. *Let Λ be the standard triangular lattice, and choose k so that $k^4 = \frac{-2}{ma_\Lambda}$ for some nonzero integer m. Then if m is of the form $3n-1$ or $3n+1$ the Fatou set of $\wp'_{k\Lambda}$ is nonempty.*

To illustrate Theorem 6.10, consider Figure 1. In this graph we use Mathematica [25] to draw the Julia and Fatou set of $\wp'_{k\Lambda}$ with k chosen so that $k^4 = \frac{-2}{ma_\Lambda}$, with $m = -1$ and Λ is the standard triangular lattice. Thus we have a superattracting 2-cycle $\{a_{k\Lambda}, -a_{k\Lambda}\}$. The Fatou set is colored blue, and the Julia set is yellow. The points of the 2-cycle, at $z \approx \pm 1.532 + 0i$, are shown as red dots, and a period parallelogram is also displayed for reference. For this lattice, we have $g_3(k\Omega) \approx -2.348$.

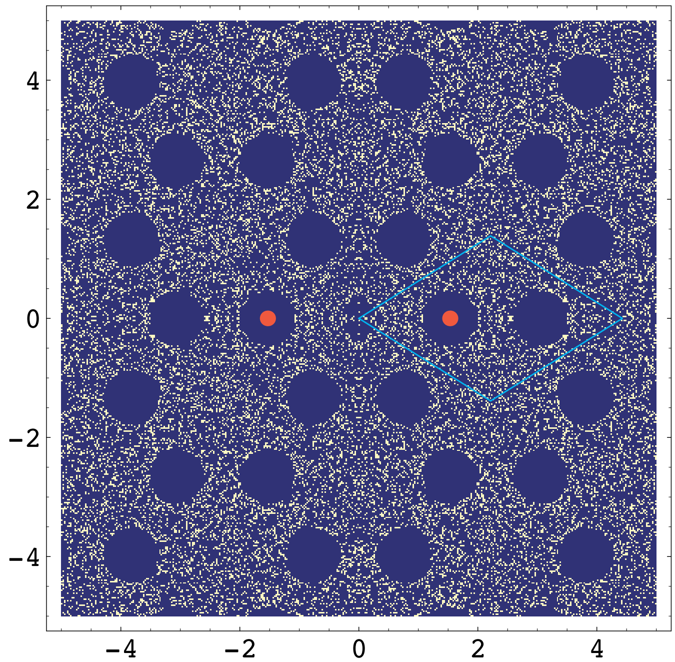

FIGURE 1. Example with a superattracting 2-cycle.

We note that the sign of k influences the orientation of the lattice. If k^4 is positive, then two of the values of k are real, one positive and one negative, and the other two values are pure imaginary, with one positive and one negative. When k is real, the lattice $k\Lambda$ is triangular in the horizontal orientation, and when k is pure imaginary, the lattice $k\Lambda$ is triangular in the vertical orientation.

If k^4 is negative then the values of k are complex; two lie on the line $y = x$ and two on the line $y = -x$. For such values of k, the lattice $k\Lambda$ is no longer a real lattice. One such example is shown is Figure 2, where we are in the setting of Theorem 6.9, and we have chosen k such that $k^4 = \frac{-2}{ma_\Lambda}$ and $m = 1$. In this case, we have two superattracting fixed points. Note for future reference that this lattice has $g_3(k\Omega) \approx -2.348i$.

The method of the last few results has been to start with a lattice Ω and choose a k value so that $k\Omega$ maps a critical point $a_{k\Omega}$ to another critical point, resulting in superattracting fixed points or superattracting two-cycles. The next proposition shows that it is impossible to construct superattracting cycles of length $n > 2$ in this way.

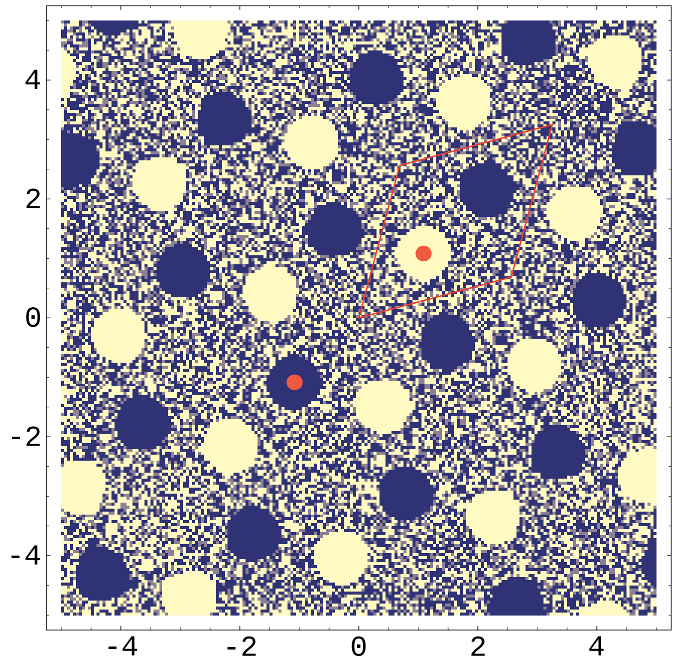

FIGURE 2. Example with two superattracting fixed points.

Proposition 6.12. *Let Ω be a triangular lattice with critical points t_1 and t_2. Suppose $\wp'_\Omega(t_1) = t_2$. If $t_1 \cong t_2$ (mod Ω), t_2 is a superattracting fixed point. If t_1 and t_2 are not congruent, t_2 and $-t_2$ form a superattracting $2-cycle$.*

Proof. Case 1: $t_1 \cong t_2$ (mod Ω). We know that $\wp'_\Omega(t_1) = t_2$. Further, by the congruence of t_1 and t_2, $\wp'_\Omega(t_2) = \wp'_\Omega(t_1) = t_2$. Thus t_2 is a superattracting fixed point.

Case 2: $t_1 \not\cong t_2$ (mod Ω). Because we are dealing with a triangular lattice, we know that there are only two congruence classes of critical points. From Equation 4.2, we also know that $\wp'_\Omega(-t_1) = -\wp'_\Omega(t_1)$; thus t_1 and $-t_1$ do not map to the same point, and therefore must be in different congruence classes. Since there are only two congruence classes of critical points, and since both t_2 and $-t_1$ are not congruent to t_1, we see that $t_2 \cong -t_1$ (mod Ω).

Again, we know that $\wp'_\Omega(t_1) = t_2$. Further, by the congruence of $-t_1$ and t_2, $\wp'_\Omega(t_2) = \wp'_\Omega(-t_1) = -\wp'_\Omega(t_1) = -t_2$. By Corollary 4.2, we also have $\wp'_\Omega(-t_2) = -\wp'_\Omega(t_2) = t_2$. Thus we have a superattracting 2-cycle. $\square$

For example, Proposition 6.12 says that we can't have a superattracting three cycle that contains two critical points.

7. Parameter Space

Next, we study the parameter space for the elliptic functions $\wp'_\Omega$ for triangular lattices Ω. The theory of holomorphic families was introduced by Mañé, Sad, and Sullivan [24], refined by McMullen [26], generalized to the setting of meromorphic maps with finite singular set by Keen and Kotus [19], and discussed for the Weierstrass elliptic function in Hawkins and Koss [14], and Hawkins and Look [16].

We begin with the definition of parameter space for triangular lattices.

Definition 7.1. *Given $g_2 = 0$, we define $g_3 -$ space to be the set of points $g_3 \in \mathbb{C} \setminus \{0\}$ which represent the triangular lattice Γ determined by $g_2 = 0$ and g_3, and therefore the function $\wp_\Gamma$.*

Let $\Gamma(0, g_3)$ denote the lattice determined by $g_2 = 0$ and $g_3 \in \mathbb{C}\setminus\{0\}$. Then the map:

$$F : \mathbb{C} \setminus \{0\} \times \mathbb{C} \to \mathbb{C}_\infty$$

given by $F(g_3, z) = \wp'_{\Gamma(0,g_3)}(z)$ is holomorphic in g_3 and meromorphic in z. This defines a holomorphic family of meromorphic functions; we say that the holomorphic family of meromorphic maps parametrized over a complex manifold M is *reduced* if for all triangular lattices $\Gamma \neq \Omega$ in M, $\wp'_\Gamma$ and $\wp'_\Omega$ are not conformally conjugate. In the current setting, g_3-space is not reduced; the next results distinguish the symmetries of parameter space arising from conjugacy from the symmetries that occur for other reasons.

We first describe the possible conformal conjugacies that can occur between two maps $\wp'_\Omega$ and $\wp'_\Gamma$ when Ω and Γ are triangular lattices.

Lemma 7.2. *If $\wp'_\Gamma$ is conformally conjugate to $\wp'_\Omega$ via a Mobius map ϕ, then $\phi(z) = az$ and $\phi(\Gamma) = \Omega$.*

Proof. Assume that $\phi(z) = (az + b)/(cz + d)$ and $\phi \circ \wp'_\Gamma(z) = \wp'_\Omega \circ \phi(z)$ for all $z \in \mathbb{C}$. Then $\phi(\infty) = \infty$ (since neither $\wp'_\Omega$ nor $\wp'_\Gamma$ is defined precisely at that point); that is, ϕ takes $\mathbb{C}$ onto $\mathbb{C}$. This means that $c = 0$ and ϕ is affine. Therefore $\phi(z) = az + b$ with $a \neq 0$; we have that $\phi(0) = b \in \Omega$ since it must be a pole of $\wp'_\Omega$.

Since critical points must be mapped to critical points under ϕ, the critical values are mapped to critical values as well. The critical values of $\wp'_\Gamma$ are $c_\Gamma = \sqrt{-g_3(\Gamma)}$ and $-c_\Gamma$, and the critical values of $\wp'_\Lambda$ are

$c_\Omega = \sqrt{-g_3(\Omega)}$ and $-c_\Omega$. First, if the critical values are mapped in the order $\phi(c_\Gamma) = c_\Omega$ and $\phi(-c_\Gamma) = -c_\Omega$, then

$$ac_\Gamma + b = c_\Omega,$$

$$-ac_\Gamma + b = -c_\Omega,$$

and therefore $b = 0$. If the critical values are mapped with the opposite pairing then $b = 0$ as well. Therefore $b = 0$, $\phi(z) = az$, and ϕ induces a group isomorphism between $\mathbb{C}/\Gamma$ and $\mathbb{C}/\Omega$; in particular $a\Gamma = \Omega$. □

Rotating a lattice by $e^{i\pi/2}$ results in conformal conjugacy.

Theorem 7.3. *If Ω is any triangular lattice and $i\Omega = \Gamma$, then $\wp_\Gamma$ and $\wp_\Omega$ are conformally conjugate via the map $\phi(z) = iz$. Further, $g_3(\Gamma) = -g_3(\Omega)$. If $\{z_1, ..., z_n\}$ is a cycle of length n with multiplier β under $\wp'_\Omega$ then $\{iz_1, ..., iz_n\}$ is a cycle of length n with multiplier β under $\wp'_{i\Omega}$.*

Proof. We need to show that $\phi \circ \wp'_\Omega(z) = \wp'_\Gamma \circ \phi(z)$. From Proposition 4.1, $\wp'_\Gamma(iz) = \wp'_{i\Omega}(iz) = i^{-3}\wp'_\Omega(z) = i\wp'_\Omega(z)$. Also from Proposition 4.1, we have that $g_3(\Gamma) = g_3(i\Omega) = i^{-6}g_3(\Omega) = -g_3(\Omega)$.

Now assume that $z_1, ..., z_n$ is a cycle of length n under $\wp'_\Omega$. From Proposition 4.1, for $1 \leq j \leq n$ we have

$$\wp'_{i\Omega}(iz_j) = \frac{1}{i^3}\wp'_\Omega(z_j) = i\wp'_\Omega(z_j) = iz_{j+1}.$$

Therefore $\{iz_1, ..., iz_n\}$ is a cycle of length n under $\wp'_{i\Omega}$.

Also, from Proposition 4.1, we have $\wp''_{i\Omega}(iz) = i^{-4}\wp''_\Omega(z) = \wp''_\Omega(z)$, so the cycles have the same multiplier. □

Although each $g_3 \in \mathbb{C} \setminus \{0\}$ corresponds to a unique lattice, the conjugacy given in Theorem 7.3 leads to $e^{\pi i}$ rotational symmetry in g_3-space coming from conformal conjugacy of the mappings. Therefore g_3-space is not a reduced space.

Theorem 7.4. *For triangular lattices $\Gamma_1 \neq \Gamma_2$, $\wp'_{\Gamma_1}$ is conformally conjugate to $\wp'_{\Gamma_2}$ if and only if $\Gamma_1 = \pm i\Gamma_2$.*

Proof. ($\Leftarrow$): By Theorem 7.3 we have that if $\Gamma_2 = e^{\pi i/2}\Gamma_1$, then $\wp'_{\Gamma_1}$ is conformally conjugate to $\wp'_{\Gamma_2}$.

($\Rightarrow$): Suppose that $\wp'_{\Gamma_1}$ is conformally conjugate to $\wp'_{\Gamma_2}$. Then the conjugating map is of the form $\phi(z) = az$ by Lemma 7.2, and we write $\Gamma_2 = a\Gamma_1$. It follows from Proposition 4.1 that $g_3(\Gamma_2) = a^{-6}g_3(\Gamma_1)$. The critical values for a triangular lattice always satisfy $c_1 = -c_2 = \sqrt{-g_3}$, by Corollary 4.4. Then $\phi(c_{\Gamma_1}) = ac_{\Gamma_1} = \pm c_{\Gamma_2}$. But

$$ac_{\Gamma_1} = a\sqrt{-g_3(\Gamma_1)} = a\sqrt{-g_3(\frac{1}{a}\Gamma_2))} = a\sqrt{-a^6 g_3(\Gamma_2)} = a^4 c_{\Gamma_2},$$

and $a^4 = \pm 1$.

If $a^4 = -1$ then $a = e^{m\pi i/4}$ for $m = 1, 3, 5$, or 7. Then $e^{m\pi i/4}\wp'_{\Gamma_1}(z) = \wp'_{e^{m\pi i/4}\Gamma_1}(e^{m\pi i/4}z)$. However, using Proposition 4.1,

$$\wp'_{e^{m\pi i/4}\Gamma_1}(e^{m\pi i/4}z) = \frac{1}{(e^{m\pi i/4})^3}\wp'_{\Gamma_1}(z),$$

which would imply that $(e^{m\pi i/4})^4 = 1$, a contradiction.

If $a = \pm 1$ then $\Gamma_2 = \Gamma_1$. The only cases remaining are when $a^2 = -1$, which gives the result.

□

Corollary 7.5. *For triangular lattices, the sector of g_3-space such that*

$$-\frac{\pi}{2} < Arg(g_3) \leq \frac{\pi}{2}$$

is a reduced holomorphic family of meromorphic maps.

Proof. By Proposition 4.1, $g_3(\Gamma_1) = g_3(i\Gamma_2) = i^{-6}g_3(\Gamma) = -g_3(\Gamma_2)$. □

In Figure 3 we have drawn a portion of g_3-space centered at the origin using Mathematica. We color each value in the g_3-plane according to the behavior of the stationary point a_Ω for $\wp'_{\Omega(0,g_3)}$. We know that the behavior of the orbits of the critical poionts a_Ω and $-a_\Omega$ are related by Proposition 6.2, so it suffices to investigate the orbit of a_Ω. Red points are values of g_3 where a_Ω is drawn to a fixed point; blue points are values of g_3 where a_Ω is drawn to a $2-cycle$; and green points are values of g_3 where a_Ω is drawn to a $3-cycle$. If no cycle was ascertained by Mathematica, then g_3 is colored orange. For reference, we have also labeled the g_3 values in Figure 3 that correspond to the examples shown in Figures 1 and 2.

We observe the $e^{\pi i}$ rotational symmetry in g_3-space predicted by Corollary 7.5. However, we also notice $e^{\pi i/2}$ rotational symmetry of shape, but not color, that does not relate to conformal conjugacy. Our next few results explain this symmetry.

First, we show that for $k = e^{\pi i/4}$, the Fatou set of $\wp'_{k\Omega}$ is the Fatou set of $\wp'_\Omega$ rotated by k.

Theorem 7.6. *Let Ω be any triangular lattice. Then $e^{\pi i/4}F(\wp'_\Omega) = F(\wp'_{e^{\pi i/4}\Omega})$ and $g_3(e^{\pi i/4}\Omega) = ig_3(\Omega)$.*

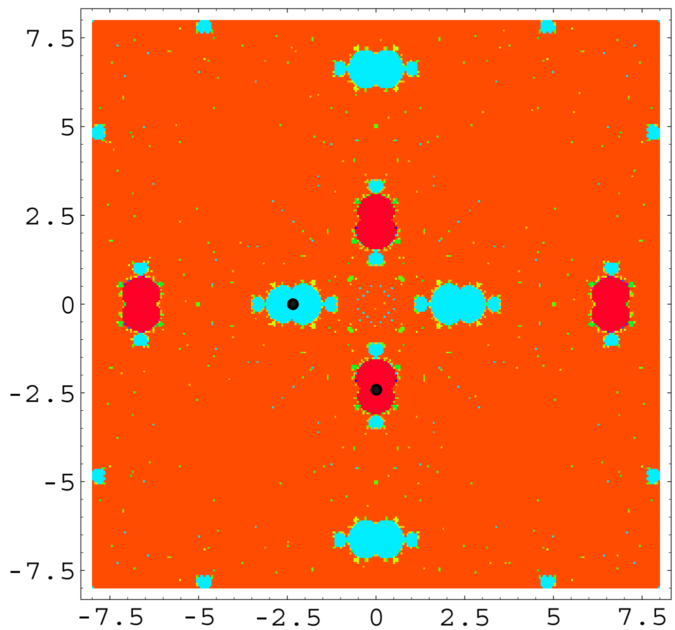

FIGURE 3. g_3-space showing $g_3 = -2.348$ and $g_3 = -2.348i$

Proof. Using Proposition 4.1, we have $\wp'_{e^{\pi i/4}\Omega}(e^{\pi i/4}z) = e^{-3\pi i/4}\wp'_\Omega(z) = -e^{\pi i/4}\wp'_\Omega(z)$, and thus $(\wp'_{e^{\pi i/4}\Omega})^n(e^{\pi i/4}z) = (-1)^n e^{\pi i/4}(\wp'_\Omega)^n(z)$. If U is a neighborhood of z such that $\{(\wp'_\Omega)^n(U)\}$ forms a normal family, then $V = e^{i\pi/4}U$ is a neighborhood of $e^{i\pi/4}z$ such that $\{(\wp'_{e^{\pi i/4}\Omega})^n(e^{\pi i/4}V)\}$ forms a normal family.

Proposition 4.1 gives that $g_3(e^{\pi i/4}\Omega) = ig_3(\Omega)$. □

In fact, we can say much more about the behavior of periodic orbits. For example, we see that under rotation by $e^{i\pi/2}$ in g_3-space, red regions become blue, meaning that all 1-cycles become 2-cycles. However, some blue regions rotate to red regions, while other blue regions remain blue under rotation. We show a close up of this phenomenon in Figures 4 and 5. The next three theorems explain what happens to periodic orbits as we rotate by $e^{i\pi/2}$ in g_3-space.

We begin by showing that cycles of odd length n turn into cycles of length $2n$ when the lattice is rotated by $e^{i\pi/4}$.

Theorem 7.7. *If $\{z_j\}_{j=1}^n$ is a cycle of odd length under $\wp'_\Omega$ and $k = e^{i\pi/4}$, then $\{(-1)^{j+1}kz_{j(mod n)}\}_{j=1}^{2n}$ is a cycle of length $2n$ under $\wp'_{k\Omega}$. Further, the $2n$-cycle has the same classification as the n-cycle.*

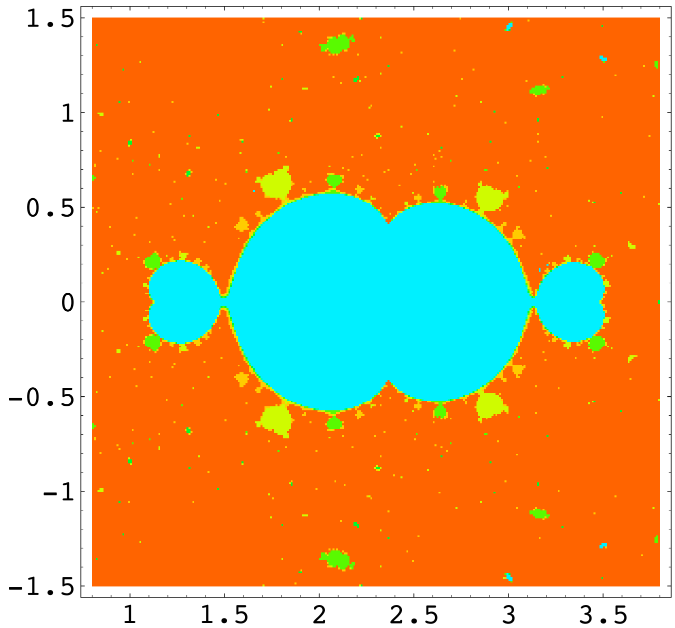

FIGURE 4. Region of g_3-space near $g_3 = 2.348$

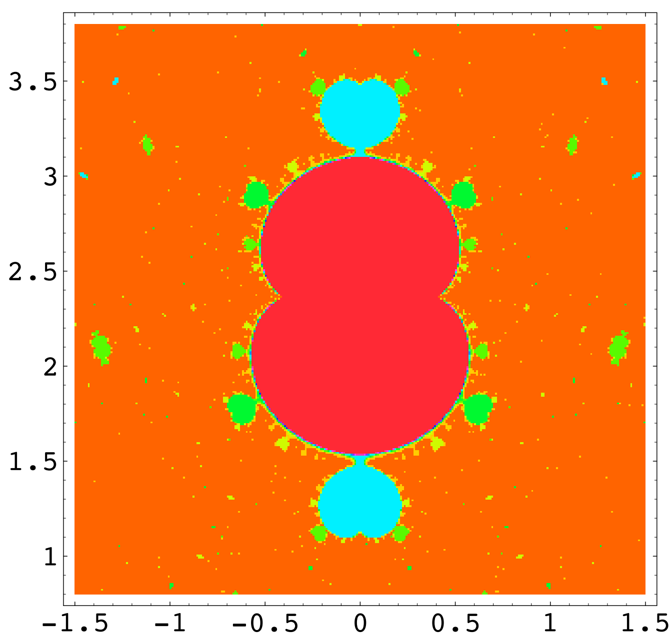

FIGURE 5. Region of g_3-space near $g_3 = 2.348i$

Proof. Let $k = e^{\pi i/4}$. Assume that $\{z_j\}_{j=1}^n$ is a cycle of odd length under $\wp'_\Omega$. From Lemma 4.1, $\wp'_{k\Omega}(kz_j) = e^{-3i\pi/4} z_j = -kz_{j+1}$ and $\wp'_{k\Omega}(-kz_j) = -e^{-3i\pi/4} z_j = kz_{j+1}$. Using these, we see that under $\wp'_{k\Omega}$,

$$kz_1 \to -kz_2 \to ... \to kz_n \to -kz_1 \to ... \to -kz_n \to kz_1.$$

Since $z_1 \neq z_j$ for $2 \leq j \leq n$, we have $kz_1 \neq kz_j$; we also know that $z_1 \neq -z_1$, and so $kz_1 \neq -kz_1$. However, we still must verify that $kz_1 \neq -kz_j$ for $2 \leq j \leq n$ to ensure that our new cycle under $\wp'_{k\Omega}$ is in fact of length $2n$. Assume not, so that $kz_1 = -kz_j$ for some $2 \leq j \leq n$. Then $z_1 = -z_j$, so that under $\wp'_\Omega$, $z_{j-1} \to -z_1$ and $-z_{j-1} \to -z_j = z_1$. Then our original cycle is $\{z_1, z_2, ..., z_{j-1}, -z_1, ..., -z_{j-1}\}$. But the length of this cycle is given by $2(j-1)$, which is even. We therefore have a contradiction, and our new cycle has length $2n$.

To see that the n-cycle and the $2n$-cycle have the same classification, we first examine $|[(\wp'_\Omega)^n]'(z_1)|$. Using the chain rule, we see that

$$[(\wp'_\Omega)^n]'(z_1) = \prod_{k=0}^{n-1} \wp''_\Omega((\wp'_\Omega)^k(z_1)).$$

Next, we consider $|[(\wp'_{k\Omega})^{2n}]'(kz_1)|$, whose value determines the classification of the $2n$-cycle. We again use the chain rule to see that

$$[(\wp'_{k\Omega})^{2n}]'(kz_1) = \wp''_{k\Omega}((\wp'_{k\Omega})^{2n-1}(kz_1))\cdot\wp''_{k\Omega}((\wp'_{k\Omega})^{2n-1}(kz_1))\cdot ... \cdot\wp''_{k\Omega}(kz_1)$$

From Proposition 4.1, we see that $(\wp'_{k\Omega})^m(kz_1) = (-1)^m k\wp'_\Omega(z_1)$ for any integer m, and that $\wp''_{k\Omega}((-1)^m kz) = -\wp''_\Omega(z)$. Then

$$\wp''_{k\Omega}((\wp'_{k\Omega})^{2n-1}(kz_1)) \cdot \wp''_{k\Omega}((\wp'_{k\Omega})^{2n-2}(kz_1)) \cdot ... \cdot \wp''_{k\Omega}(kz_1) =$$
$$\wp''_{k\Omega}((-1)^{2n-1}k{\wp'_\Omega}^{2n-1}(z_1)) \cdot \wp''_{k\Omega}((-1)^{2n-2}k{\wp'_\Omega}^{2n-2}(z_1)) \cdots \wp''_{k\Omega}(kz_1) =$$
$$-\wp''_\Omega({\wp'_\Omega}^{2n-1}(z_1)) \cdot -\wp''_\Omega({\wp'_\Omega}^{2n-2}(z_1)) \cdot ... \cdot -\wp''_\Omega(z_1).$$

Finally, we recall that z_1 has period n under $\wp'_\Omega$. Because of this, ${\wp'_\Omega}^{2n-s}(z_1) = {\wp'_\Omega}^{n-s}(z_1)$. Therefore, $|[(\wp'_{k\Omega})^{2n}]'(kz_1)| = |[(\wp'_\Omega)^n]'(z_1)|^2$, and so the $2n$-cycle formed by kz_1 under $\wp'_{k\Omega}$ has the same classification as the n-cycle formed by z_1 under $\wp'_\Omega$. □

For example, suppose $\{z_1, z_2, z_3\}$ is a 3-cycle under $\wp'_\Omega$. Theorem 7.7 then says that $\{kz_1, -kz_2, kz_3, -kz_1, kz_2, -kz_3\}$ is a 6-cycle under $\wp'_{k\Omega}$, and that the 6-cycle has the same classification as the 3-cycle.

From Lemma 4.1, $g_3(k\Omega) = ig_3(\Omega)$; thus, we have shown that any region corresponding to a cycle with odd length n will become a region corresponding to a cycle of length $2n$ under rotation by $e^{i\pi/2}$. Specifically, regions corresponding to fixed points will become regions corresponding to 2-cycles. Next we have a similar result involving n-cycles of even length satisfying a certain structure and rotation by $e^{i\pi/2}$.

Theorem 7.8. *If* $\{(-1)^{j+1}z_{j(mod n)}\}_{j=1}^{2n}$ *with* n *odd is a cycle under* $\wp'_\Omega$ *and* $k = e^{i\pi/4}$*, then* $\{kz_j\}_{j=1}^{n}$ *and* $\{-kz_j\}_{j=1}^{n}$ *are cycles of odd length under* $\wp'_{k\Omega}$*. Further, the classification of the* n*-cycle under* $\wp'_{k\Omega}$ *is the same as the classification of the* $2n$*-cycle under* $\wp'_\Omega$*.*

Proof. Let $k = e^{i\pi/4}$. Assume that $\{(-1)^{j+1}z_{j(mod n)}\}_{j=1}^{2n}$ with n odd is a cycle under $\wp'_\Omega$. From Lemma 4.1, $\wp'_{k\Omega}(kz_j) = e^{-3i\pi/4}\wp'_\Omega(z_j) = -k\wp'_\Omega(z_j) = (-k)(-z_{j+1}) = kz_{j+1}$. Thus, under $\wp'_{k\Omega}$, $kz_1 \to kz_2 \to \dots \to kz_n \to kz_1$. We know that $z_1 \neq z_j$ for all $2 \leq j \leq n$, so it is also true that $kz_1 \neq kz_j$ for all $2 \leq j \leq n$. Thus $\{kz_j\}_{j=1}^{n}$ is a cycle under $\wp'_{k\Omega}$, as is $\{-kz_j\}_{j=1}^{n}$.

The proof that the n-cycle under $\wp'_{k\Omega}$ has the same classification as the $2n$-cycle under $\wp'_\Omega$ follows from the same reasoning as in the proof of Theorem 7.7. □

For example, if $\{z_1, -z_2, z_3, -z_1, z_2, -z_3\}$ is a cycle under $\wp'_\Omega$, then $\{kz_1, kz_2, kz_3\}$ is a cycle under $\wp'_{k\Omega}$, and both cycles have the same classification.

Observe that Theorems 7.7 and 7.8 are very closely related. If we begin with a g_3 value that corresponds to a lattice with a cycle of odd length n, then $e^{i\pi/2}g_3$ corresponds to a lattice with a cycle of length $2n$; another rotation by $e^{i\pi/2}$ gives a g_3 value whose lattice again has an n cycle. Our theorems have also given us a clear relationship between the cycles generated by the different g_3 values. Also note that we are talking about a specific type of n-cycle in Theorem 7.8. Our next theorem deals with all other cycles of even length.

Theorem 7.9. *Suppose* $\{z_j\}_{j=1}^{m}$ *is a cycle of even length* m *under* $\wp'_\Omega$ *and does not have the form* $\{(-1)^{j+1}z_{j(mod n)}\}_{j=1}^{2n}$ *with* n *odd. If* $k = e^{i\pi/4}$*, then* $\{(-1)^{j+1}kz_j\}_{j=1}^{m}$ *is a cycle of length* m *under* $\wp'_{k\Omega}$*. Further, the classification of the* m*-cycle under* $\wp'_{k\Omega}$ *is the same as the classification of the* m*-cycle under* $\wp'_\Omega$*.*

Proof. Using Proposition 4.1, we know that $\wp'_{k\Omega}(kz_j) = -kz_{j+1}$ and $\wp'_{k\Omega}(-kz_j) = kz_{j+1}$. Thus, under $\wp'_{k\Omega}$,

$$kz_1 \to -kz_2 \to \dots \to kz_{m-1} \to -kz_m \to kz_1.$$

We know that $kz_1 \neq kz_j$, so to ensure that $\{(-1)^{j+1}kz_j\}_{j=1}^{m}$ is a cycle of length m under $\wp'_{k\Omega}$, we must check that $kz_1 \neq -kz_j$ for j even. Assume not, so that $z_1 = -z_j$ for some even $2 \leq j \leq m$; note that $j-1$ is odd. Then $\wp'_\Omega(z_{j-1}) = -z_1$, and our original cycle has the form $\{z_1, \dots, z_{j-1}, -z_1, \dots, -z_{j-1}\}$ and has length $2(j-1)$. If m is not divisible by an odd number, the contradiction is immediate. On the other hand, if m is divisible by an odd number, then our cycle has the form $\{(-1)^{j+1}z_{j(mod n)}\}_{j=1}^{2n}$ with n odd, which contradicts our assumption.

To show that $|[(\wp'_\Omega)^m]'(z_1)| = |[(\wp'_{k\Omega})^m]'(kz_1)|$, we use that $(\wp'_{k\Omega})^n(kz_1) = (-1)^n k\wp'_\Omega(z_1)$ and $\wp''_{k\Omega}((-1)^n kz) = -\wp''_\Omega(z)$ for any integer n. □

Theorem 7.9 shows that if we begin with a g_3 value that gives a cycle of even length m that does not have the specific form given in Theorem 7.8, then $e^{i\pi/2}g_3$ again gives us an m-cycle. That is, unless an even cycle has a certain form, it remains the same length when g_3-space is rotated by $e^{i\pi/2}$.

We note the appearance of Mandelbrot-like sets in g_3 parameter space in the sense introduced by Douady and Hubbard [7], McMullen [26], and extended to quadratic-like Weierstrass elliptic functions by Hawkins and Look [16]. We do not present any results here; they are a further subject of study by the authors.

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