# Linear Regression Models
# P8111

Lecture 04

Jeff Goldsmith
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THE DEPARTMENT OF BIOSTATISTICS
Columbia University MAILMAN SCHOOL OF PUBLIC HEALTH

# Today's lecture

- Simple Linear Regression
- Least Squares Estimation

# Regression modeling

- Want to use predictors to learn about the outcome distribution, particularly conditional expected value.
- Formulate the problem parametrically

$$E(y \mid x) = f(x; \beta) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots$$

- (Note that other useful quantities, like covariance and correlation, tell you about the joint distribution of $y$ and $x$)

# Covariance and Correlation

# Simple linear regression

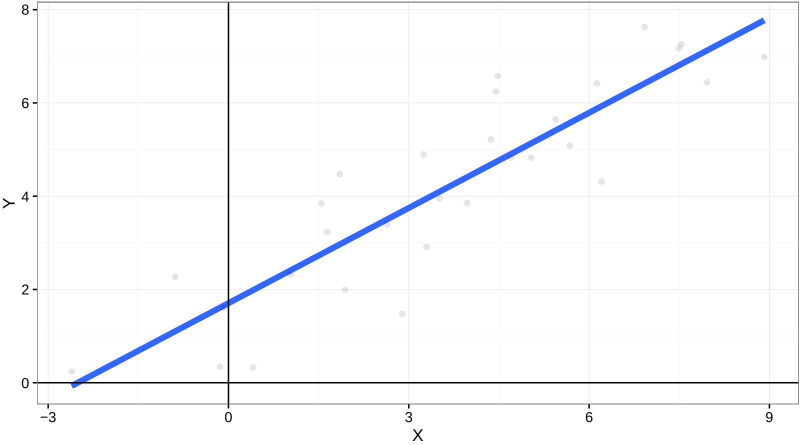
- Linear models are a special case of all regression models; simple linear regression is the simplest place to start
- Only one predictor:

$$E(y \mid x) = f(x; \beta) = \beta_0 + \beta_1 x_1$$

- Useful to note that $x_0 = 1$ (implicit definition)
- Somehow, estimate $\beta_0, \beta_1$ using observed data.

# Coefficient interpretation

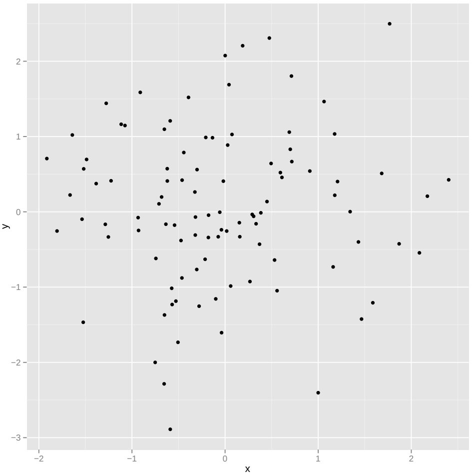
# Coefficient interpretation



# Look at the data

- Plot the data (using `ggplot` ...)
- Do the data look like the assumed model?
- Should you be concerned about outliers?
- Define what you expect to see before fitting any model.

# Look at the data



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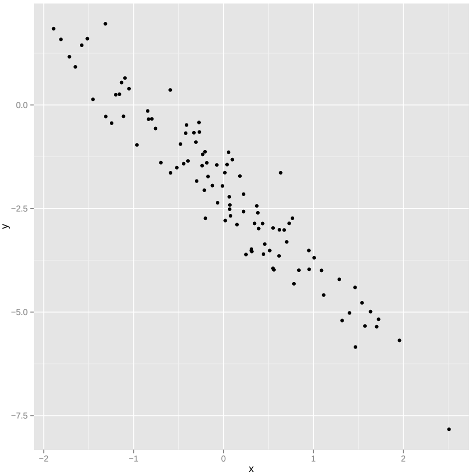

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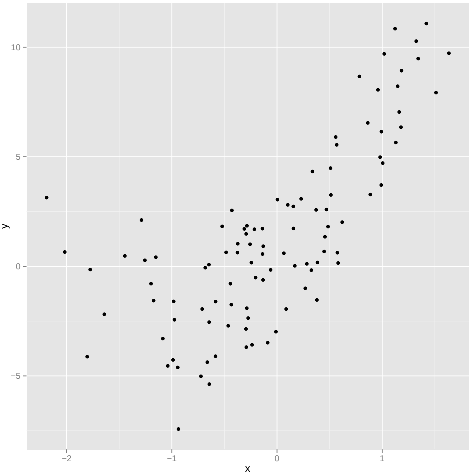

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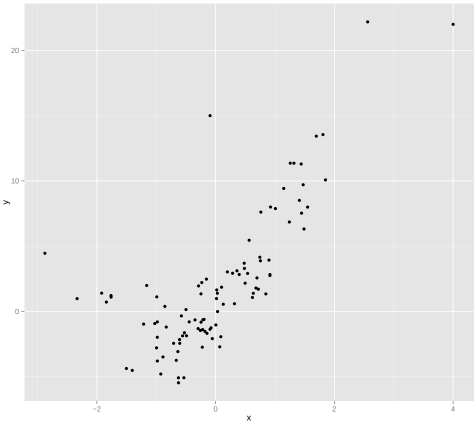


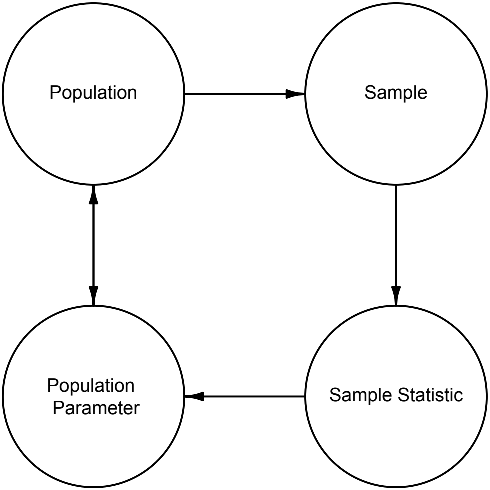
# Least squares estimation

- Observe data $(y_i, x_i)$ for subjects $1, \ldots, n$. Want to estimate $\beta_0, \beta_1$ in the model

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i;\ \epsilon_i \overset{iid}{\sim} (0, \sigma^2)$$

- Note the assumptions on the variance:
    - $E(\epsilon \mid x) = E(\epsilon) = 0$
    - Constant variance
    - Independence
    - [Normally distributed is not needed for least squares, but is needed for inference]

# Circle of Life



# Least squares estimation

- Recall that for a single sample $y_i, i \in 1, \ldots, n$, the sample mean $\hat{\mu}_y$ minimizes the sum of squared deviations.

# Least squares estimation

- Find $\hat{\beta}_0$.

# Least squares estimation

- Now find $\hat{\beta}_1$.

# Note about correlation

$$\rho = \frac{cov(x,y)}{\sqrt{var(x)var(y)}}; \qquad \beta_1 = \frac{cov(x,y)}{var(x)}$$

# R does exactly what we now expect

```
> linmod = lm(y~x, data = data)
> summary(linmod)

Call:
lm(formula = y ~ x, data = data)

Residuals:
    Min      1Q  Median      3Q     Max
-1.5202 -0.5050 -0.2297  0.5753  1.8534

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  2.08743    0.22958   9.092 7.53e-10 ***
x            0.61396    0.05415  11.338 5.61e-12 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.8084 on 28 degrees of freedom
Multiple R-squared:  0.8211,Adjusted R-squared:  0.8148
F-statistic: 128.6 on 1 and 28 DF,  p-value: 5.612e-12
```

# R does exactly what we now expect

```
> tidy(linmod)
         term  estimate  std.error statistic      p.value
1 (Intercept) 2.0874344 0.22958105  9.092364 7.529711e-10
2           x 0.6139621 0.05415004 11.338166 5.611585e-12
> glance(linmod)
  r.squared adj.r.squared     sigma statistic      p.value df   logLik ...
1  0.821148     0.8147604 0.8084399   128.554 5.611585e-12  2 -35.1538 ...
>
>
> beta1 = with(data, sum((x - mean(x))*(y - mean(y))) / sum((x - mean(x))^2))
> beta0 = with(data, mean(y) - beta1*mean(x))
> c(beta0, beta1)
[1] 2.0874344 0.6139621
```

# Note on interpretation of $\beta_0$

Recall $\beta_0 = E(y|x = 0)$

- This often makes no sense in context
- "Centering" $x$ can be useful: $x^* = x - \bar{x}$
- Center by mean, median, minimum, etc
- Effect of centering on slope:

# Note on interpretation of $\beta_0, \beta_1$

- The interpretations are sensitive to the scale of the outcome and predictors (in reasonable ways)
- You can't get a better model fit by rescaling variables

# R example

```
> data = mutate(data, x.cen = x - mean(x), x2 = x*2)
> linmod.cen = lm(y ~ x.cen, data = data)
> tidy(linmod.cen)
         term  estimate  std.error statistic      p.value
1 (Intercept) 4.0811993 0.14760027  27.65035 7.172437e-22
2       x.cen 0.6139621 0.05415004  11.33817 5.611585e-12
```

# R example

```
> linmod.x2 = lm(y ~ x2, data = data)
> tidy(linmod.x2)
         term  estimate  std.error statistic      p.value
1 (Intercept) 2.0874344 0.22958105  9.092364 7.529711e-10
2          x2 0.3069811 0.02707502 11.338166 5.611585e-12
```

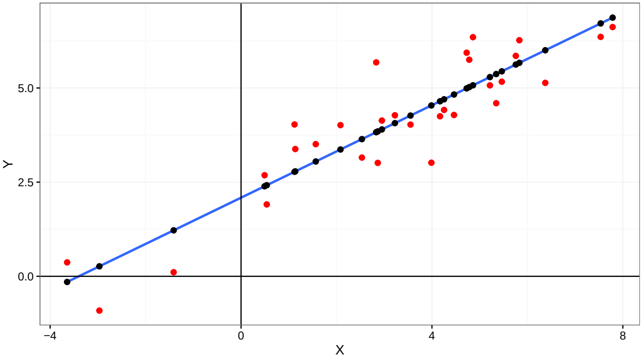
# Least squares notes and foreshadowing

- Didn't have to choose to minimize squares – could minimize absolute value, for instance.
- Least squares estimates turn out to be a "good idea" – unbiased, BLUE.
- Later we'll see about maximum likelihood as well.

# Geometric interpretation of least squares

Least squares minimizes the sum of squared vertical distances between observed and estimated $y$'s:

$$\underset{\beta_0,\,\beta_1}{min} \sum_{i=1}^{I} (y_i - (\beta_0 + \beta_1 x_i))^2$$



# Least squares in regression generally

Broadly speaking, in regression we often are concerned with minimizing

$$E[f(x) + \epsilon - \hat{f}(x)]^2$$

by choosing a "good" $\hat{f}$. For a given $\hat{f}$ this decomposes into

$$E[f(x) - \hat{f}(x)]^2 + \mathrm{Var}(\epsilon)$$

- Some variance isn't explainable (we just don't know how much)
- Focus on getting the left component right
- Minimizing squared error for *unseen* data is the real goal

# Today's big ideas

- Simple linear regression – model and interpretation
- Least squares estimation

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- Suggested reading: Faraway Ch 1, 2.1; ISLR 3.1