Linear Regression Models
P8111

Lecture 06

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Today's lecture

- Multiple Linear Regression
    - Assumptions
    - Interpretation
    - Some models

Motivation

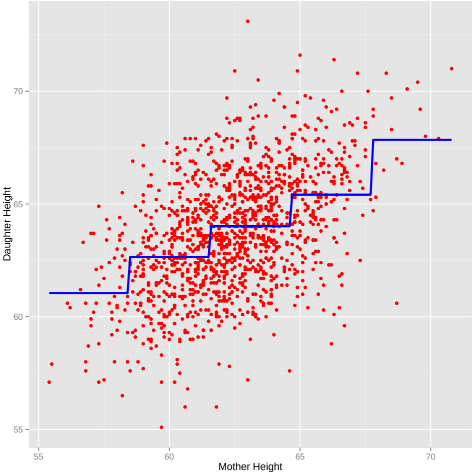
Most applications involve more that one covariate – if more than one thing can influence an outcome, you need multiple linear regression.

- Improved description of $y|\boldsymbol{x}$
- More accurate estimates and predictions
- Allow testing of multiple effects
- Includes multiple predictor types

Why not bin all predictors?

- Divide x_i into k_i bins
- Stratify data based on inclusion in bins across x's
- Find mean of the y_i in each category
- Possibly a reasonable non-parametric model

Why not bin all predictors?



Why not bin all predictors?

- More predictors = more bins
- If each x has 5 bins, you have 5^p overall categories
- May not have enough data to estimate distribution in each category
- Curse of dimensionality is a problem in a lot of non-parametric statistics

Multiple linear regression model

- Observe data $(y_i, x_{i1}, \ldots, x_{ip})$ for subjects $1, \ldots, n$. Want to estimate $\beta_0, \beta_1, \ldots, \beta_p$ in the model

$$y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_1 x_{ip} + \epsilon_i;\ \epsilon_i \overset{iid}{\sim} (0, \sigma^2)$$

- Assumptions (residuals have mean zero, constant variance, are independent) are as in SLR
- Impose linearity which (as in the SLR) is a big assumption
- Our primary interest will be $E(y|\boldsymbol{x})$
- Eventually estimate model parameters using least squares

Predictor types

- Continuous
- Categorical
- Ordinal

Interpretation of coefficients

$$\beta_0 = E(y|x_1 = 0, \ldots, x = 0)$$

- Centering some of the x's may make this more interpretable

Interpretation of coefficients

$\beta_1 =$

Example with two predictors

Suppose we want to regress weight on age and sex.

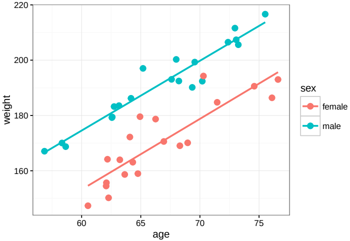
- Model is $y_i = \beta_0 + \beta_1 x_{i,age} + \beta_2 x_{i,sex} + \epsilon_i$
- Age is continuous starting with age 0; sex is binary, coded so that $x_{i,sex} = 0$ for men and $x_{i,sex} = 1$ for women
  - In your dataset, sex should be a `factor` variable ...

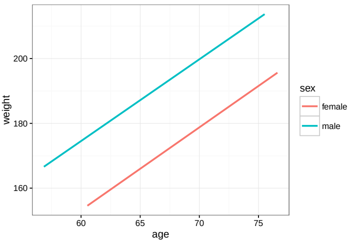
Example with two predictors

$$\beta_1 =$$

$$\beta_2 =$$

Example with two predictors





Example: MLR

```
> summary(data.mlr)
      age             sex          weight     
 Min.   :56.86   female:20   Min.   :147.3  
 1st Qu.:62.71   male  :20   1st Qu.:168.3  
 Median :65.72               Median :181.4  
 Mean   :66.70               Mean   :180.9  
 3rd Qu.:70.23               3rd Qu.:193.0  
 Max.   :76.60               Max.   :216.6  
```

Example: MLR

```
> linmod = lm(weight ~ age + sex, data = data.mlr)
> tidy(linmod)
         term  estimate  std.error   statistic      p.value
1 (Intercept)  1.060460 12.2596341  0.08650011 9.315353e-01
2         age  2.537768  0.1828033 13.88250888 3.022310e-16
3     sexmale 21.116049  1.8470731 11.43216717 1.056992e-13
```

Example: MLR

```
> summary(linmod)

Call:
lm(formula = weight ~ age + sex, data = data.mlr)

Residuals:
    Min      1Q  Median      3Q     Max 
-8.8987 -3.2152 -0.2969  2.3688 14.8074 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   1.0605    12.2596   0.087    0.932    
age           2.5378     0.1828  13.883 3.02e-16 ***
sexmale      21.1160     1.8471  11.432 1.06e-13 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 5.841 on 37 degrees of freedom
Multiple R-squared:  0.8977,Adjusted R-squared:  0.8921
F-statistic: 162.3 on 2 and 37 DF,  p-value: < 2.2e-16
```

Example: MLR

```
> head(data.mlr)
Source: local data frame [6 x 3]

       age    sex   weight
     (dbl) (fctr)    (dbl)
1 62.58799   male 179.4342
2 65.18893   male 197.0306
3 73.06852   male 207.3838
4 56.85860   male 167.0692
5 69.56368   male 199.3080
6 67.99770   male 200.2703
> model.matrix(linmod) %>% head
  (Intercept)      age sexmale
1           1 62.58799       1
2           1 65.18893       1
3           1 73.06852       1
4           1 56.85860       1
5           1 69.56368       1
6           1 67.99770       1
```

Example: MLR

```
> tail(data.mlr)
Source: local data frame [6 x 3]

       age    sex   weight
     (dbl) (fctr)    (dbl)
1 64.75572 female 158.9645
2 63.64315 female 158.6567
3 64.08004 female 172.2003
4 64.32532 female 163.0857
5 68.96513 female 170.1063
6 64.93602 female 179.5558
> model.matrix(linmod) %>% tail
   (Intercept)      age sexmale
35           1 64.75572       0
36           1 63.64315       0
37           1 64.08004       0
38           1 64.32532       0
39           1 68.96513       0
40           1 64.93602       0
```

Omitted variable bias

What happens if we ignore x_2 and fit the simple linear regression:

$$y_i = \beta_0^* + \beta_1^* x_{i,1} + \epsilon_i^*$$

Does $\beta_1^* = \beta_1$? Does "total" association equal "partial" association?

Omitted variable bias

Omitted variable bias

There are two conditions under which $E(\beta_1^*) = \beta_1$:

- The omitted variable is unrelated to the outcome
- The omitted variable is uncorrelated with the retained variable

Still only two predictors

Suppose we think that the effect of age on weight is different for men and women. How might we approach this problem?

- Separate models?
- Interactions?

Interpretation of coefficients

Example: Interactions

```
> linmod = lm(weight ~ age * sex, data = data.mlr)
> tidy(linmod)
         term   estimate  std.error statistic      p.value
1 (Intercept) -33.529162 16.3317392 -2.053006 4.739377e-02
2         age   2.714362  0.2468607 10.995522 4.636445e-13
3     sexmale -63.357026 22.7687183 -2.782635 8.532471e-03
4 age:sexmale   2.049536  0.3412012  6.006825 6.805174e-07
```

Example: Interactions

```
> head(data.mlr)
Source: local data frame [6 x 3]

       age    sex   weight
     (dbl) (fctr)    (dbl)
1 62.24128   male 199.0986
2 67.10186   male 220.4382
3 60.98623   male 198.6198
4 75.57168   male 263.4126
5 67.97705   male 221.7642
6 61.07719   male 190.6024
> model.matrix(linmod) %>% head
  (Intercept)      age sexmale age:sexmale
1           1 62.24128       1    62.24128
2           1 67.10186       1    67.10186
3           1 60.98623       1    60.98623
4           1 75.57168       1    75.57168
5           1 67.97705       1    67.97705
6           1 61.07719       1    61.07719
```

Example: Interactions

```
> tail(data.mlr)
Source: local data frame [6 x 3]

       age    sex   weight
     (dbl) (fctr)    (dbl)
1 57.73764 female 116.8223
2 63.51003 female 140.5238
3 63.63426 female 136.4259
4 65.64412 female 144.1169
5 72.60015 female 161.9464
6 70.57905 female 152.9105
> model.matrix(linmod) %>% tail
   (Intercept)      age sexmale age:sexmale
35           1 57.73764       0           0
36           1 63.51003       0           0
37           1 63.63426       0           0
38           1 65.64412       0           0
39           1 72.60015       0           0
40           1 70.57905       0           0
```

Categorical predictors

- Assume X is a categorical / nominal / factor variable with k levels
- With only one categorical X, we have the classic one-way ANOVA design
- Can't use a single predictor with levels $1, 2, \ldots, K$ – this has the wrong interpretation
- Need to create *indicator* or *dummy* variables

Indicator variables

- Let x be a categorical variable with k levels (e.g. with $k = 3$ "low", "med", "high").
- Choose one group as the baseline (e.g. "low")
- Create $(k-1)$ binary terms to include in the model:

$$\begin{aligned} x_{\text{med},i} &= I(x_i = \text{“med”}) \\ x_{\text{high},i} &= I(x_i = \text{“high”}) \end{aligned}$$

- For a model with no additional predictors, pose the model

$$y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_{k-1} x_{i,k-1} + \epsilon_i$$

  and estimate parameters using least squares
- Note distinction between *predictors* and *terms*

Categorical predictor design matrix

ANOVA model interpretation

Using the model $y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_{k-1} x_{i,k-1} + \epsilon_i$, interpret
$\beta_0 =$

$\beta_1 =$

Equivalent model

Define the model $y_i = \beta_1 x_{i1} + \ldots + \beta_k x_{i,k} + \epsilon_i$ where there are indicators for each possible group

$\beta_1 =$

$\beta_2 =$

Example for categorical predictor

Suppose you want to compare the effect of placebo, exercise and a drug on blood pressure. You set up a trial to do this and gather data $y_i, treatment_i$ on n subjects.

- Analyze results using the model

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i$$

where x_i1 indicates that subject i exercised and x_i2 indicates that subject i received medication.

Example: categorical predictor

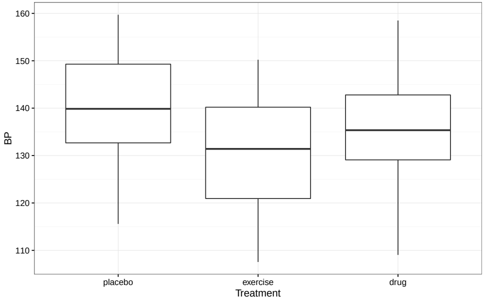
```
> ## load data
> load("BPDat.RDA")
>
> ## see what we've loaded
> head(BP)
  x1       x2
1  1 149.5939
2  1 155.5605
3  1 129.5920
4  1 149.3057
5  1 139.2455
6  1 120.3280
> summary(BP)
       x1          x2
 Min.   :1   Min.   :107.5
 1st Qu.:1   1st Qu.:128.0
 Median :2   Median :136.8
 Mean   :2   Mean   :169.9
 3rd Qu.:3   3rd Qu.:144.8
 Max.   :3   Max.   :999.0
```

Example: categorical predictor

```
> ## tidy data
> BP = BP %>% rename(Treatment = x1, BP = x2) %>%
+   mutate(Treatment = factor(Treatment, levels = 1:3,
+                             labels = c("placebo", "exercise", "drug"))) %>%
+   filter(BP != 999)
>
> summary(BP)
   Treatment        BP
 placebo :47   Min.   :107.5
 exercise:47   1st Qu.:127.0
 drug    :50   Median :136.7
               Mean   :135.3
               3rd Qu.:143.5
               Max.   :159.7
>
> BP %>% group_by(Treatment) %>% summarize(n = n(),
+                                          group_mean = mean(BP),
+                                          group_median = median(BP))
Source: local data frame [3 x 4]

  Treatment     n group_mean group_median
     (fctr) (int)      (dbl)        (dbl)
1   placebo    47   140.3368     139.8598
2  exercise    47   130.6135     131.4055
3      drug    50   135.0942     135.3504
```

Example: categorical predictor



Example: categorical predictor

$$bp_i = \beta_0 + \beta_1 tx_{\text{exer},i} + \beta_2 tx_{\text{drug},i} + \epsilon_i$$

```
> lm(BP ~ Treatment, data = BP) %>% tidy
              term   estimate std.error statistic       p.value
1       (Intercept) 140.336772  1.647753 85.168558 3.906601e-123
2 Treatmentexercise  -9.723234  2.330275 -4.172569  5.240892e-05
3     Treatmentdrug  -5.242587  2.295055 -2.284297  2.384739e-02
>
>
> lm(BP ~ Treatment, data = BP) %>% model.matrix %>% head
  (Intercept) Treatmentexercise Treatmentdrug
1           1                 0             0
2           1                 0             0
3           1                 0             0
4           1                 0             0
5           1                 0             0
6           1                 0             0
```

Example: releveling categorical predictor

$$bp_i = \beta_0 + \beta_1 tx_{\text{plac},i} + \beta_2 tx_{\text{drug},i} + \epsilon_i$$

```
> BP %>% mutate(Treatment = relevel(Treatment, ref = "exercise")) %>%
+   lm(BP ~ Treatment, data = .) %>%
+   tidy
             term   estimate std.error statistic       p.value
1     (Intercept) 130.613538  1.647753 79.267654 7.929548e-119
2 Treatmentplacebo   9.723234  2.330275  4.172569  5.240892e-05
3    Treatmentdrug   4.480647  2.295055  1.952305  5.288319e-02
```

Example: no intercept

$$bp_i = \beta_1 tx_{\text{exer},i} + \beta_2 tx_{\text{plac},i} + \beta_3 tx_{\text{drug},i} + \epsilon_i$$

```
> lm(BP ~ 0 + Treatment, data = BP) %>% tidy
              term estimate std.error statistic       p.value
1  Treatmentplacebo 140.3368  1.647753  85.16856 3.906601e-123
2 Treatmentexercise 130.6135  1.647753  79.26765 7.929548e-119
3     Treatmentdrug 135.0942  1.597556  84.56303 1.048075e-122
>
>
> BP %>% group_by(Treatment) %>% summarize(n = n(),
+                                          group_mean = mean(BP),
+                                          group_median = median(BP))
Source: local data frame [3 x 4]

  Treatment     n group_mean group_median
     (fctr) (int)      (dbl)        (dbl)
1   placebo    47   140.3368     139.8598
2  exercise    47   130.6135     131.4055
3      drug    50   135.0942     135.3504
```

Today's big ideas

- Multiple linear regression models, interpretation, interactions, categorical predictors

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- Suggested reading: Faraway Ch 2.2 - 2.3; ISLR 3.2