Linear Regression Models
P8111

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Lecture 09

10:00 am → 5:00pm

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Midterm!!

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3B!!!

THE DEPARTMENT OF BIOSTATISTICS
Columbia University MAILMAN SCHOOL OF PUBLIC HEALTH

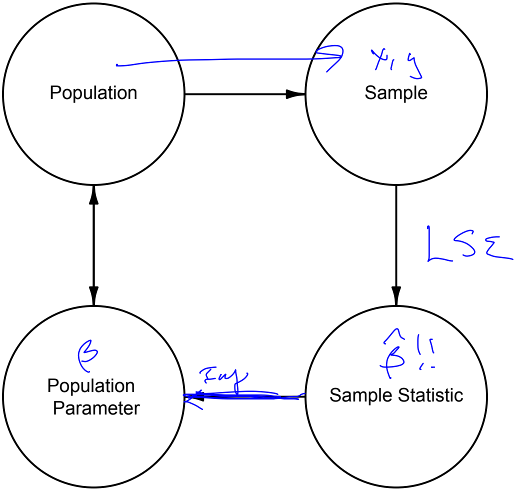
Today's Lecture

MLR : LSE $\rightarrow \hat{\beta} = (X^T X)^{-1} X^T y$

$\hookrightarrow y = X\beta + \epsilon$

- Sampling distribution of $\hat{\beta}$
- Hypothesis tests for individual coefficients
- Global tests

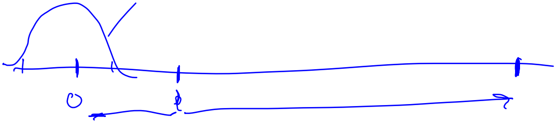
Circle of Life



Statistical inference

- We have LSEs $\hat{\beta}_0, \hat{\beta}_1, \ldots$; we want to know what this tells us about $\beta_0, \beta_1, \ldots$.
- Two basic tools are confidence intervals and hypothesis tests
  - Confidence intervals provide a plausible range of values for the parameter of interest based on the observed data
  - Hypothesis tests ask how probable are the data we gathered under a null hypothesis about the data generating distribution

A quick word about p-values



P-values ...

- Are not universally adored
  - Compares data vs null (usually no effect) rather than testing whether data are consistent with your real hypothesis
  - Often misinterpreted (“probability the null is true”)
- Can get people in trouble
  - Especially when misinterpreted
- Are still the default tool for inference

"why most published Research findings are false"

Motivation

Recall the MLB data:

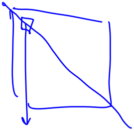
```
> setwd("~/Desktop")
> download.file("http://www.openintro.org/stat/data/mlb11.RData", destfile = "mlb11.RData")
> load("mlb11.RData")
>
> mlb11 %>% tbl_df
Source: local data frame [30 x 12]

                  team  runs at_bats  hits homeruns bat_avg strikeouts stolen_bases  wins new_
                (fctr) (int)   (int) (int)    (int)   (dbl)      (int)        (int) (int)
1        Texas Rangers   855    5659  1599      210   0.283        930          143    96
2       Boston Red Sox   875    5710  1600      203   0.280       1108          102    90
3       Detroit Tigers   787    5563  1540      169   0.277       1143           49    95
4   Kansas City Royals   730    5672  1560      129   0.275       1006          153    71
5  St. Louis Cardinals   762    5532  1513      162   0.273        978           57    90
6        New York Mets   718    5600  1477      108   0.264       1085          130    77
7     New York Yankees   867    5518  1452      222   0.263       1138          147    97
8    Milwaukee Brewers   721    5447  1422      185   0.261       1083           94    96
9     Colorado Rockies   735    5544  1429      163   0.258       1201          118    73
10      Houston Astros   615    5598  1442       95   0.258       1164          118    56
..                 ...   ...     ...   ...      ...     ...        ...          ...   ...
Variables not shown: new_slug (dbl), new_obs (dbl)
```

Motivation

$$\hat{\beta} = (X^T X)^{-1} X^T y$$

$$\hat{var}(\hat{\beta}) = \hat{\sigma}^2 (X^T X)^{-1}$$



$\hat{var}(\hat{\beta}_0)$

st err $= \sqrt{\ }$

std ERR T-stat pval

```
Call:
lm(formula = runs ~ at_bats + hits + homeruns + stolen_bases,
    data = mlb11)
 ...
Coefficients:
             Estimate
(Intercept)  581.2110
at_bats       -0.2023
hits           0.6974
homeruns       1.2535
stolen_bases   0.5230
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 26.85 on 25 degrees of freedom
Multiple R-squared: 0.9087,Adjusted R-squared: 0.894
F-statistic: 62.17 on 4 and 25 DF,  p-value: 1.26e-12
```

Motivation

$\hat{\beta}$

- Can we say anything about whether the effect of `stolen_bases` is "significant" after adjusting for other variables? $H_0: \hat{\beta}_{SB} = 0$
- Can we compare this model to a model with only `hits` and `homeruns`?

$y_i = \beta_0 + \beta_1 \, at + \beta_2 \, hits + \beta_3 \, hR + \beta_4 \, SB + \epsilon$

vs

$\hookrightarrow y_i = \beta_0 + \beta_1 \, hits + \beta_2 \, hR + \epsilon$

$H_0: \beta_4 = 0$ then $H_0: \beta_1 = 0$

$H_0: \beta_4 = \beta_1 = 0$

Sampling distribution

$\hat{\beta} = (X^T X)^{-1} X^T y$ $\qquad$ $\hat{\beta} \sim (\beta, \sigma^2 (X^T X)^{-1})$

$y \sim N(X\beta, \sigma^2 I)$

If our usual assumptions are satisfied and $\epsilon \overset{iid}{\sim} \mathrm{N}\left[0, \sigma^2\right]$ then

$$\hat{\boldsymbol{\beta}} \sim \mathrm{N}\left[\boldsymbol{\beta}, \sigma^2 (\boldsymbol{X}^T \boldsymbol{X})^{-1}\right].$$

$$\hat{\beta}_j \sim \mathrm{N}\left[\boldsymbol{\beta}_j, \sigma^2 (\boldsymbol{X}^T \boldsymbol{X})^{-1}_{jj}\right].$$

- This will be used for inference.

Asymptotic distribution

Assume that

- $E(\epsilon_i|\boldsymbol{x}_i) = 0 \ \forall i$;
- $Var(\epsilon_i|\boldsymbol{x}_i) = \sigma^2 \ \forall i$;
- $n \overset{lim}{\rightarrow} \infty \ \frac{\boldsymbol{X}^T\boldsymbol{X}}{n} \rightarrow Q$ where Q is a finite non-singular matrix.

Then

$$\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \rightarrow \mathrm{N}\left[0, \sigma^2 Q^{-1}\right]$$

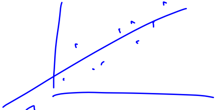
(This is essentially an extension of the central limit theorem)

Simulations exploring distributions

$n \in \{10,\ 100,\ 1000\}$

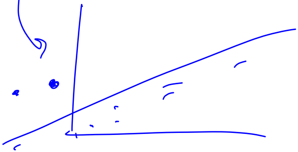
Look at SLR

$$y_i = 0 + 1x_i + \epsilon_i$$



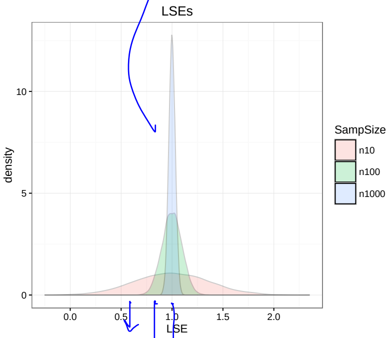
under various conditions.

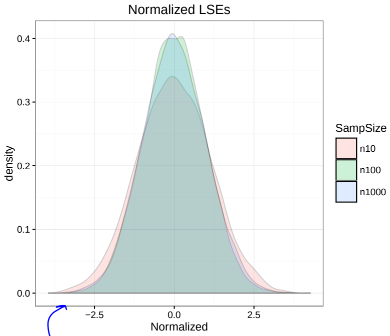
- First simulations: errors follow $\mathrm{N}\,[0, 1]$, let n vary
- Second simulations: errors follow $\frac{10}{3} * \mathrm{Bern}(.1) - \frac{1}{3}$, let n vary
- In both cases, $\epsilon \sim (0, 1)$



Normal errors

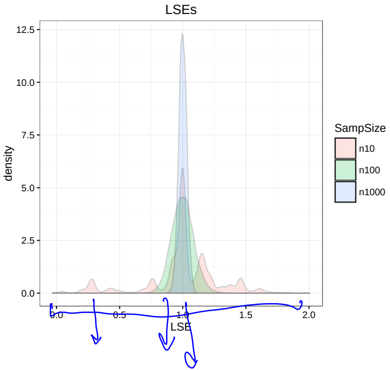
$\hat{\beta}_1$

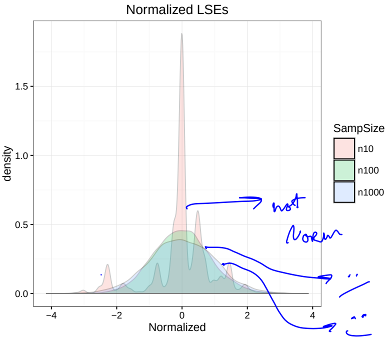




$\sqrt{n}(\hat{\beta}_1 - \beta_1)$

Non-normal errors





not Norm

$\sqrt{n}(\hat{\beta}_1 - \beta_1)$

Testing procedure

Calculate the probability of the observed data (or more extreme data) under a null hypothesis.

- Often $H_0 : \beta_1 = 0$ and $H_a : \beta_1 \neq 0$
- Set $\alpha = P$(falsely rejecting a true null hypothesis) (type I error rate) 0.05
- Calculate a test statistic assuming the null hypothesis is true
- Compute a p-value =

$$P(\text{As or more extreme test statistic}|H_0)$$

- Reject or fail to reject H_0

Testing

$$\hat{\beta}_j \sim N(\beta, \sigma^2 (X^T X)^{-1}_{jj})$$

For real data we have to estimate σ^2 as well as $\boldsymbol{\beta}$.

- Recall our estimate of the error variance is

$$\hat{\sigma^2} = \frac{RSS}{n-p-1} = \frac{\sum_i (y_i - \hat{y}_i)^2}{n-p-1}$$

- With Normally distributed errors, it can be shown that

$$(n-p-1)\frac{\hat{\sigma^2}}{\sigma^2} \sim \chi^2_{n-p-1}$$

Implication is that test statistics follow a t distribution rather than Normal with $df = n - p - 1$

Individual coefficients

For individual coefficients

- We can use the test statistic

$$T = \frac{\hat{\beta}_j - \beta_j}{\widehat{se}(\hat{\beta}_j)} = \frac{\hat{\beta}_j - \beta_j}{\sqrt{\hat{\sigma}^2 (\boldsymbol{X}^T \boldsymbol{X})^{-1}_{jj}}} \sim t_{n-p-1}$$

- For a two-sided test of size α, we reject if

$$|T| > t_{1-\alpha/2, n-p-1}$$

- The p-value gives $P(t_{n-p-1} > T_{obs} | H_0)$

Note that t is a symmetric distribution that converges to a Normal as $n - p - 1$ increases.

Example revisited

```
Call:
lm(formula = runs ~ at_bats + hits + homeruns + stolen_bases,
    data = mlb11)
...
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  581.2110   526.4063   1.104  0.28006
at_bats       -0.2023     0.1174  -1.724  0.09706 .
hits           0.6974     0.1131   6.164 1.91e-06 ***
homeruns       1.2535     0.1593   7.868 3.18e-08 ***
stolen_bases   0.5230     0.1686   3.101  0.00473 **
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 26.85 on 25 degrees of freedom
Multiple R-squared: 0.9087,Adjusted R-squared: 0.894
F-statistic: 62.17 on 4 and 25 DF,  p-value: 1.26e-12
```

Inference for linear combinations

Sometimes we are interested in making claims about $c^T\boldsymbol{\beta}$ for some c.

- Define $H_0 : c^T\boldsymbol{\beta} = c^T\boldsymbol{\beta}_0$ or $H_0 : c^T\boldsymbol{\beta} = 0$
- We can use the test statistic

$$T = \frac{c^T\hat{\boldsymbol{\beta}} - c^T\boldsymbol{\beta}_0}{\widehat{se}(c^T\hat{\boldsymbol{\beta}})} = \frac{c^T\hat{\boldsymbol{\beta}} - c^T\boldsymbol{\beta}_0}{\sqrt{\hat{\sigma}^2 c^T(\boldsymbol{X}^T\boldsymbol{X})^{-1}c}}$$

- This test statistic is asymptotically Normally distributed
- For a two-sided test of size α, we reject if

$$|T| > z_{1-\alpha/2}$$

Inference about multiple coefficients

Our model contains multiple parameters; often we want to perform multiple tests:

$$\begin{aligned} H_{01} : \beta_1 &= 0 \\ H_{02} : \beta_2 &= 0 \\ \vdots &= \vdots \\ H_{0k} : \beta_k &= 0 \end{aligned}$$

where each test has a size of α

- For any individual test, $P(\text{reject } H_{0i} | H_{0i}) = \alpha$

Inference about multiple coefficients

What about

$$P(\text{reject at least one } H_{0i} | \text{all } H_{0i} \text{ are true}) = \alpha$$

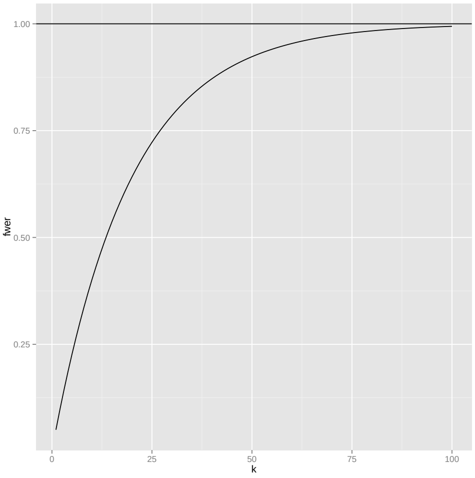
Family-wise error rate

To calculate the FWER

- First note $P(\text{no rejections}|\text{all } H_{0i} \text{ are true}) = (1-\alpha)^k$
- It follows that $P(\text{at least one rejection}|\text{all } H_{0i} \text{ are true}) = 1-(1-\alpha)^k$
- Further,

$$\begin{aligned} FWER = 1-(1-\alpha)^k &= 1-\left(1-\frac{k\alpha}{k}\right)^k \\ &\approx 1-exp(1-k\alpha) \\ &\approx 1-(1-k\alpha) \\ &= k\alpha \end{aligned}$$

Family-wise error rate



Addressing multiple comparisons

Three general approaches

- Do nothing in a reasonable way
  - Define comparisons and expectations ahead of time
  - Don't trust scientifically implausible results
  - Don't over-emphasize isolated findings
- Correct for multiple comparisons
  - Often, use the Bonferroni correction and use $\alpha_i = \alpha/k$ for each test
  - Thanks to the Bonferroni inequality, this gives an overall $FWER \leq \alpha$
  - Control false discovery rate
- Use a global test

Global tests

Compare a smaller "null" model to a larger "alternative" model

- Smaller model must be nested in the larger model
- That is, the smaller model must be a special case of the larger model
- For both models, the *RSS* gives a general idea about how well the model is fitting
- In particular, something like

$$\frac{RSS_S - RSS_L}{RSS_L}$$

  compares the relative *RSS* of the models

Nested models

- These models are nested:

$$\begin{aligned} \text{Smaller} &= \text{Regression of } Y \text{ on } X_1 \\ \text{Larger} &= \text{Regression of } Y \text{ on } X_1, X_2, X_3, X_4 \end{aligned}$$

- These models are not:

$$\begin{aligned} \text{Smaller} &= \text{Regression of } Y \text{ on } X_2 \\ \text{Larger} &= \text{Regression of } Y \text{ on } X_1, X_3 \end{aligned}$$

Global F tests

- Compute the test statistic

$$F_{obs} = \frac{(RSS_S - RSS_L)/(df_S - df_L)}{RSS_L/df_L}$$

- If H_0 (the null model) is true, then $F_{obs} \sim F_{df_S - df_L, df_L}$
- Note $df_s = n - p_S - 1$ and $df_L = n - p_L - 1$
- We reject the null hypothesis if the p-value is above α, where

$$\text{p-value} = P(F_{df_S - df_L, df_L} > F_{obs})$$

Global F tests

There are a couple of important special cases for the F test

- The null model contains the intercept only
  - When people say ANOVA, this is often what they mean (although all F tests are based on an analysis of variance)
- The null model and the alternative model differ only by one term
  - Gives a way of testing for a single coefficient
  - Turns out to be equivalent to a two-sided t-test: $t^2_{df_L} \sim F_{1,df_L}$

MLB data

You can test multiple coefficient simultaneously using the *F* test

```
> linmod.null1 = lm(runs ~ hits + homeruns, data = mlb11)
> anova(linmod.null1, linmod)
Analysis of Variance Table

Model 1: runs ~ hits + homeruns
Model 2: runs ~ at_bats + hits + homeruns + stolen_bases
  Res.Df   RSS Df Sum of Sq      F   Pr(>F)
1     27 27128
2     25 18020  2    9107.8 6.3178 0.006015 **
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

MLB data

The *F* test is equivalent to the *t* test when there's only one parameter of interest

```
> linmod.null2 = lm(runs ~ at_bats + hits + homeruns, data = mlb11)
> anova(linmod.null2, linmod)
Analysis of Variance Table

Model 1: runs ~ at_bats + hits + homeruns
Model 2: runs ~ at_bats + hits + homeruns + stolen_bases
  Res.Df   RSS Df Sum of Sq     F   Pr(>F)
1     26 24953
2     25 18020  1    6932.7 9.618 0.004728 **
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

MLB data

By default, R's `summary` function compares to an intercept-only null model

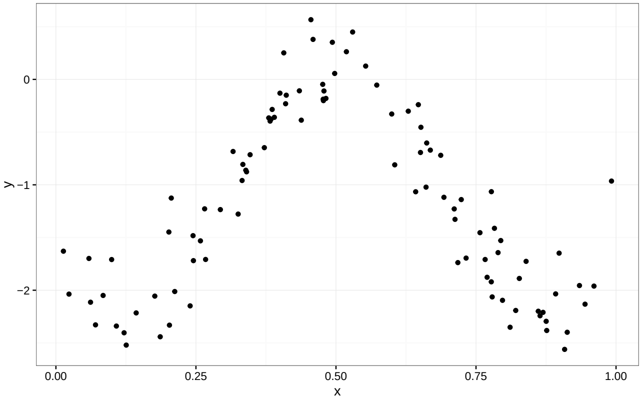
```
> linmod.null3 = lm(runs ~ 1, data = mlb11)
> anova(linmod.null3, linmod)
Analysis of Variance Table

Model 1: runs ~ 1
Model 2: runs ~ at_bats + hits + homeruns + stolen_bases
  Res.Df    RSS Df Sum of Sq      F   Pr(>F)
1     29 197281
2     25  18020  4    179261 62.174 1.26e-12 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

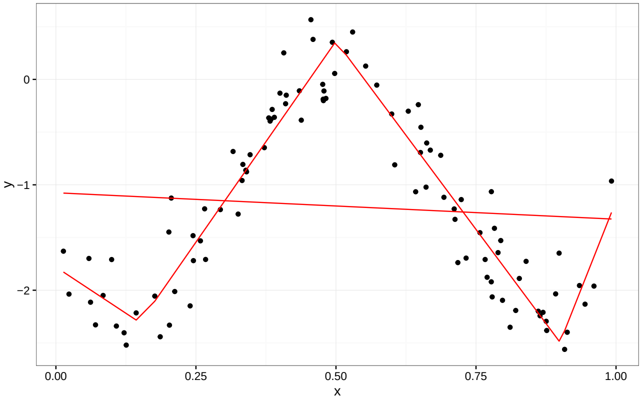
Test for "linearity"

- To test more flexible vs less flexible approaches to non-linearity, we can often use global tests
  - Polynomials and piecewise linear models have "linear" associations as nested model; B-splines don't
- Global F tests can be pretty useful here

Testing for linearity



Testing for linearity

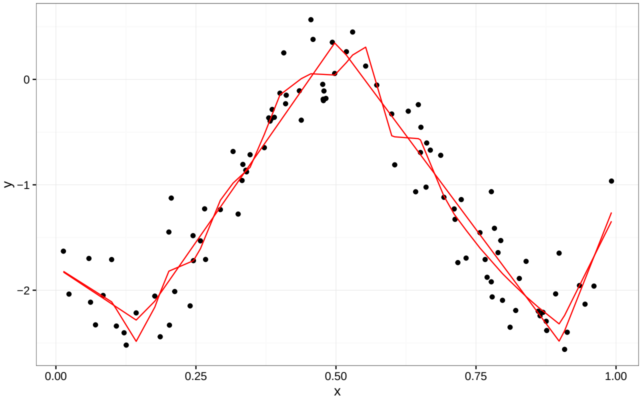


Testing linearity

```
> piecewise.underfit = lm(y ~ x, data = data.nonlin)
> piecewise.fit = lm(y ~ x + spline_15 + spline_5 + spline_9, data = data.nonlin)
> anova(piecewise.underfit, piecewise.fit)
Analysis of Variance Table

Model 1: y ~ x
Model 2: y ~ x + spline_15 + spline_5 + spline_9
  Res.Df    RSS Df Sum of Sq     F    Pr(>F)
1     98 73.444
2     95  8.240  3    65.205 250.6 < 2.2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Testing comparing twenty polynomials to four



Testing comparing twenty polynomials to four

```
> piecewise.overfit = lm(y ~ x + spline_1 + spline_15 + spline_2 + spline_25 + spline_3 + spli
+                         spline_4 + spline_45 + spline_5 + spline_55 + spline_6 + spline_65
+                         spline_7 + spline_75 + spline_8 + spline_85 + spline_9,
+                       data = data.nonlin)
> anova(piecewise.fit, piecewise.overfit)
Analysis of Variance Table

Model 1: y ~ x + spline_15 + spline_5 + spline_9
Model 2: y ~ x + spline_1 + spline_15 + spline_2 + spline_25 + spline_3 +
    spline_35 + spline_4 + spline_45 + spline_5 + spline_55 +
    spline_6 + spline_65 + spline_7 + spline_75 + spline_8 +
    spline_85 + spline_9
  Res.Df    RSS Df Sum of Sq     F Pr(>F)
1     95 8.2395
2     81 6.7862 14    1.4533 1.239 0.2645
```

Testing comparing twenty polynomials to four

```
> anova(piecewise.underfit, piecewise.fit, piecewise.overfit)
Analysis of Variance Table

Model 1: y ~ x
Model 2: y ~ x + spline_15 + spline_5 + spline_9
Model 3: y ~ x + spline_1 + spline_15 + spline_2 + spline_25 + spline_3 +
    spline_35 + spline_4 + spline_45 + spline_5 + spline_55 +
    spline_6 + spline_65 + spline_7 + spline_75 + spline_8 +
    spline_85 + spline_9
  Res.Df    RSS Df Sum of Sq       F Pr(>F)
1     98 73.444
2     95  8.240  3    65.205 259.427 <2e-16 ***
3     81  6.786 14     1.453   1.239 0.2645
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Today's big ideas

- Inference for multiple linear regression models

- Suggested reading: Faraway Ch 3.1 - 3.3