

Linear Regression Models

P8111

Lecture 10

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Today's Lecture

- ✓ ■ Review of tests
- ✓ ■ Two new tests
- ✓ ■ Confidence intervals
- > ■ Foreshadowing

~~Resampling~~
Resampling

Some review notes

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \epsilon_i$$

$\epsilon \sim (0, \sigma^2)$

β

- Do we have a test for the null $H_0 : \beta_3 = -14$ ✓
- Do we have a test for the null $H_0 : \beta_2 + \beta_3 = \pi$ ✓
- Do we have a test for the null $H_0 : \beta_2 \beta_3 = \pi$ ✗

$$\frac{\hat{\beta}_3 - (-14)}{\widehat{se}(\hat{\beta}_3)}$$

$$C = [0 \ 0 \ 1 \ 1]$$

$$\frac{C\hat{\beta} - \pi}{\widehat{se}(C\hat{\beta})}$$

Individual coefficients

For individual coefficients

$$H_0: \beta_j = 0$$

- We can use the test statistic

$$T = \frac{\hat{\beta}_j - \beta_j}{\widehat{se}(\hat{\beta}_j)} = \frac{\hat{\beta}_j - \beta_j}{\sqrt{\hat{\sigma}^2(\mathbf{X}^T \mathbf{X})_{jj}^{-1}}} \sim t_{n-p-1}$$

- For a two-sided test of size α , we reject if

$$|T| > t_{1-\alpha/2, n-p-1}$$

- The p-value gives $P(|t_{n-p-1}| > |T_{obs}| | H_0)$

Note that t is a symmetric distribution that converges to a Normal as $n - p - 1$ increases.

Inference for linear combinations

$$H_0: \beta_2 = \beta_3 \quad (\text{maybe categorical})$$
$$H_a: \beta_2 - \beta_3 = 0$$
$$c = [0 \ 0 \ 1 \ -1] \quad c: X_1 - X_2$$

Sometimes we are interested in making claims about $c^T \beta$ for some c .

- Define $H_0 : c^T \beta = c^T \beta_0$ or $H_0 : c^T \beta = 0$
- We can use the test statistic

$$T = \frac{c^T \hat{\beta} - c^T \beta}{\widehat{se}(c^T \hat{\beta})} = \frac{c^T \hat{\beta} - c^T \beta}{\sqrt{\hat{\sigma}^2 c^T (\mathbf{X}^T \mathbf{X})^{-1} c}}$$

$$c = [0 \ 1 \ 1]$$
$$c \left[\begin{array}{c} \sqrt{\text{Var}(\hat{\beta})} \\ \vdots \\ \vdots \end{array} \right] c^T$$

- This test statistic is asymptotically Normally distributed
- For a two-sided test of size α , we reject if

$$|T| > z_{1-\alpha/2}$$

$$c \widehat{\text{Var}}(\hat{\beta}) c^T$$

Global F tests

$$\underline{H_0: \beta_2 = \beta_3 = 0} \quad / \quad \text{useful for categorical}$$

- Compute the test statistic

$$F_{obs} = \frac{(RSS_S - RSS_L)/(df_S - df_L)}{RSS_L/df_L}$$

- If H_0 (the null model) is true, then $F_{obs} \sim F_{df_S - df_L, df_L}$
- Note $df_S = n - p_S - 1$ and $df_L = n - p_L - 1$
- We reject the null hypothesis if the p-value is above α , where

$$\text{p-value} = P(F_{df_S - df_L, df_L} > F_{obs})$$

Alternative global tests: the Wald test

$$\left(\frac{\hat{\beta}_j - \beta_0}{\widehat{se}(\hat{\beta}_j)} \right)^2 = t^2$$

For a vector of coefficients, we can test $H_0 : \beta = \beta_0$:

- Use the test statistic

$$W = (\hat{\beta} - \beta_0)^T [\widehat{Var}(\hat{\beta})]^{-1} (\hat{\beta} - \beta_0)$$

- Under the null, this test statistic has an asymptotic χ_p^2 distribution
- In practice, we replace $Var(\hat{\beta})$ with $\widehat{Var}(\hat{\beta})$ and use an F distribution

Alternative global tests: the Wald test

The previous test is special case of $H_0 : R\beta = R\beta_0$ for a $d \times p$ matrix R , using $R = I_{p \times p}$:

$$R = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{bmatrix} \quad R\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix}$$

- Use the test statistic

$$W = (\underbrace{R \text{Var}(\hat{\beta}) R^T}_{\text{bracketed}})^{-1} (R\hat{\beta} - R\beta_0)$$

- Under the null, this test statistic has an asymptotic χ_d^2 distribution, where

$$d = \text{rank}(\text{Var}(R\hat{\beta}))$$

- This formulation is useful for testing subsets (e.g. $H_0 : \beta_1 = \beta_2 = 0$)

Alternative global tests: the likelihood ratio test

$$H_0: \beta_2 = \beta_3 = 0$$

If we are using maximum likelihood estimation (we'll cover this soon – turns out to be least squares in MLR), we can use a LRT:

- Use the test statistics

$$\Delta = -2 \log \frac{L_0}{L_1} = -2(l_0 - l_1)$$

Likelihood for null
for alt

- This test statistic has an asymptotic χ_d^2 distribution where d is the difference in the number of parameters between the two models.
- Must compare nest models

Example: LRT

$$H_0: \beta_1 = \beta_4 = 0$$

```
> linmod = lm(runs ~ at_bats + hits + homeruns + stolen_bases, data = mlb11)
> linmod.null1 = lm(runs ~ hits + homeruns, data = mlb11)
> anova(linmod.null1, linmod)
Model 1: runs ~ hits + homeruns
Model 2: runs ~ at_bats + hits + homeruns + stolen_bases
  Res.Df  RSS Df Sum of Sq    F    Pr(>F)
1      27 27128
2      25 18020  2    9107.8  6.3178 0.006015 **
>
> ## LRT
> Delta = -2*(logLik(linmod.null1) - logLik(linmod))
> 1-pchisq(Delta, 2)
'log Lik.' 0.002163305 (df=4)
```

Confidence intervals: individual parameters

$$H_0: \beta_j = \beta_0 \Rightarrow \frac{\hat{\beta}_j - \beta_0}{\widehat{se}(\hat{\beta}_j)}$$

- A confidence interval with coverage $(1 - \alpha)$ is given by

$$\hat{\beta}_j \pm t_{1-\alpha/2, n-p-1} \widehat{se}(\hat{\beta}_j) \quad \left(\hat{\beta}_j \pm z \widehat{se}(\hat{\beta}_j) \right)$$

- Assuming all the standard assumptions hold,

$$(1 - \alpha) \text{ "="" } P(LB < \beta_j < UB)$$

$$.95 \quad 0 / 1$$

Note there is a one-to-one correspondence between this confidence interval and the hypothesis test.

$$\frac{\hat{\beta}_j - \beta_0}{\widehat{se}(\hat{\beta}_j)} < t_{\alpha, df} \Rightarrow \beta_0 \in \left(\hat{\beta}_j \pm t_{\alpha, df} \widehat{se}(\hat{\beta}_j) \right)$$

Example revisited

```
> confint(linmod)
                2.5 %      97.5 %
(Intercept) -502.9429878 1.665365e+03
at_bats      -0.4440385  3.938287e-02
hits         0.4643923  9.304364e-01
homeruns     0.9253836  1.581629e+00
stolen_bases 0.1756711  8.702772e-01
```

Confidence intervals: multiple parameters

We might want a confidence region for multiple coefficients simultaneously

- Invert Wald test for multiple coefficients – find region containing all values β_0 for which p-value from global Wald test is $> \alpha$

- Then

$$(1 - \alpha) \text{ "="" } P[\beta \in \text{region}]$$

- This region is an ellipsoid in higher dimensions; we can visualize in 2D most easily and 3D pretty well.

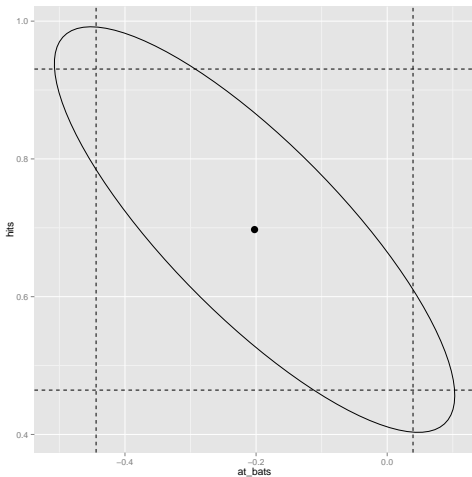
Confidence intervals: multiple parameters

```
library(ellipse)

CI.ellipse = as.data.frame(ellipse(linmod,c(2,3)))
est = as.data.frame(t(as.matrix(coef(linmod)[2:3])))

## plot the joint confidence region
ggplot(CI.ellipse, aes(x = at_bats, y = hits)) + geom_path() +
  geom_hline(yintercept = confint(linmod)[3,], linetype = 2) +
  geom_vline(xintercept = confint(linmod)[2,], linetype = 2) +
  geom_point(data = est, size = 4)
```

Confidence intervals: multiple parameters



Expected value and it's variance

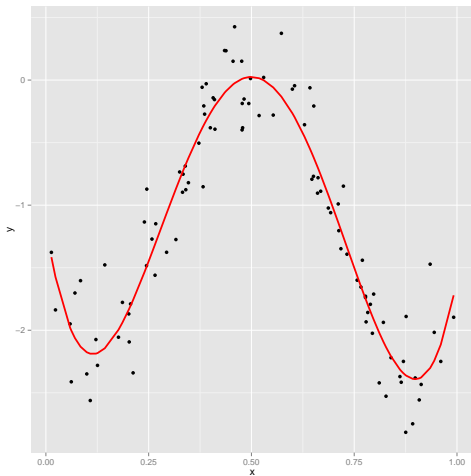
- What is $\hat{E}(y|x = x_0)$?
- How can we estimate the variance of $\hat{E}(y|x = x_0)$?

In particular, a confidence interval for $E(y|x = x_0)$ is given by

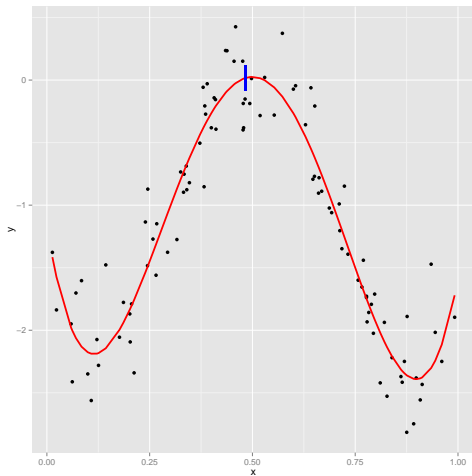
$$(\hat{y}|x = x_0) \pm t_{1-\alpha/2, n-p-1} \widehat{se}_{fit}(\hat{y}|x_0)$$

This can be estimated for any x_0 .

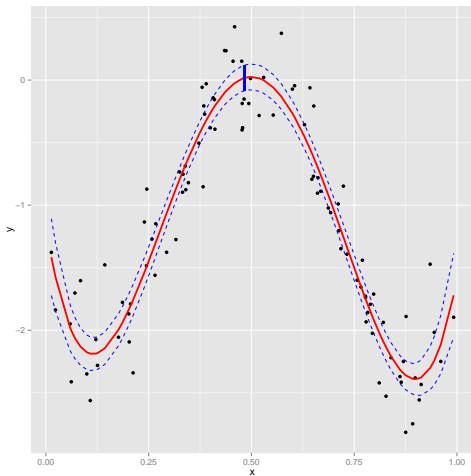
Estimated mean



Estimated mean and variance



Estimated mean and variance



Pointwise and simultaneous CIs

- Pointwise confidence intervals construct CI's at each point independently of all other points
- Implicit multiple comparisons problem
- Simultaneous intervals can be constructed so that

$$(1 - \alpha) \text{ "="" } P(f(x) \in \text{SCI})$$

Which is wider?

Predictions and prediction intervals

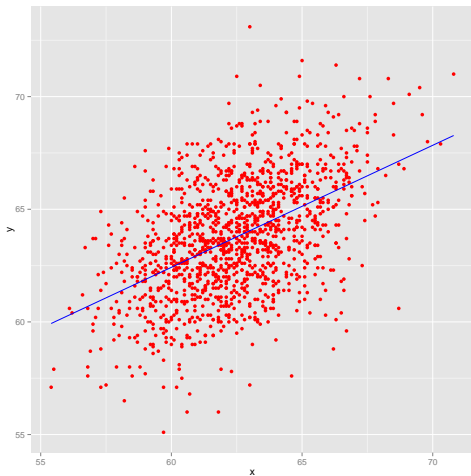
- What is the prediction value y for a given x_0
- What range would you give for the value of a new outcome?
- Two sources of variance to consider: variance in estimates and variance in outcome

A prediction interval is given by

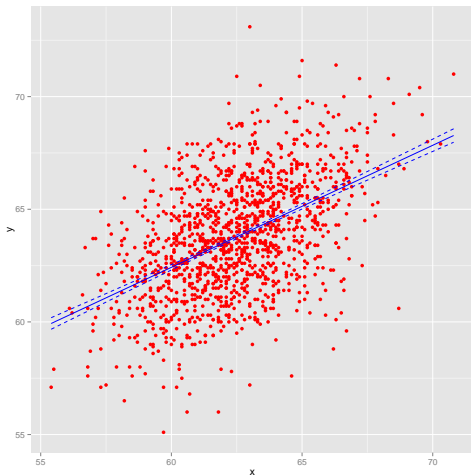
$$(\hat{y}|\mathbf{x} = \mathbf{x}_0) \pm t_{1-\alpha/2, n-p-1} \hat{se}_{pred}(\hat{y}|\mathbf{x}_0)$$

This can be estimated for any x_0 .

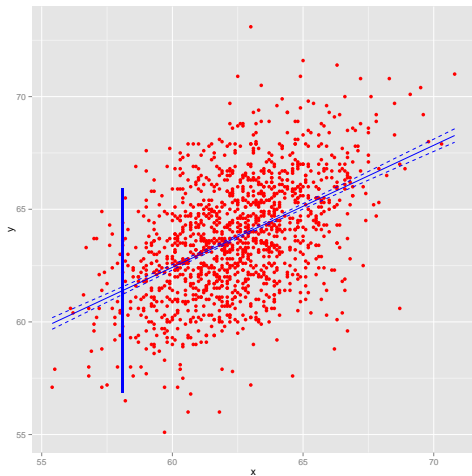
Predictions - mother/daughter height



Variance of fitted values



Prediction interval



Some things to think on

- Why are we building models?
- How should we assess models?
- What kinds of predictors should we included, and how should we decide to include them?

Some things to think on

Three general goals are

- Prediction
- Estimation of association
- Testing of associations

These goals will often not lead to the same final model.

Today's big ideas

- Inference for MLRs: global tests; confidence intervals for coefficients, predictions, functions

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- Suggested reading: Faraway Ch 3.6 - 3.9