

Linear Regression Models

P8111

Lecture 11

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Today's Lecture

$H_0: \dots$

(confidence intervals)

L_9, L_{10}

- Review of tests
- The bootstrap
- Permutation testing
- Cross validation

Individual coefficients

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \dots + \beta_p x_{ip} + \epsilon_i$$
$$\epsilon_i \sim (0, \sigma^2)$$

For individual coefficients

- We can use the test statistic $H_0: \beta_j = 0$

$$T = \frac{\hat{\beta}_j - \cancel{\beta_j}^0}{\widehat{se}(\hat{\beta}_j)} = \frac{\hat{\beta}_j - \beta_j}{\sqrt{\hat{\sigma}^2 (\mathbf{X}^T \mathbf{X})_{jj}^{-1}}} \sim t_{n-p-1}$$

- For a two-sided test of size α , we reject if

$$|T| > t_{1-\alpha/2, n-p-1}$$

- The p-value gives $P(|t_{n-p-1}| > |T_{obs}| | H_0)$

Note that t is a symmetric distribution that converges to a Normal as $n - p - 1$ increases.

Inference for linear combinations

$$c = [0 \quad 1 \quad -1] \quad c\beta = \beta_1 - \beta_2 \quad H_0: \beta_1 = \beta_2$$

Sometimes we are interested in making claims about $c^T\beta$ for some c .

- Define $H_0 : c^T\beta = c^T\beta_0$ or $H_0 : c^T\beta = 0$
- We can use the test statistic

$$\underline{T} = \frac{c^T\hat{\beta} - c^T\beta}{\widehat{se}(c^T\hat{\beta})} = \frac{c^T\hat{\beta} - c^T\beta}{\sqrt{\hat{\sigma}^2 c^T(\mathbf{X}^T\mathbf{X})^{-1}c}}$$

- This test statistic is asymptotically Normally distributed
- For a two-sided test of size α , we reject if

$$|T| > z_{1-\alpha/2}$$

Global F tests

$$H_0: \beta_2 = \beta_3 = \beta_4 = 0$$

- Compute the test statistic

$$F_{obs} = \frac{(RSS_S - RSS_L)/(df_S - df_L)}{RSS_L/df_L}$$

- If H_0 (the null model) is true, then $F_{obs} \sim F_{df_S - df_L, df_L}$
- Note $df_S = n - p_S - 1$ and $df_L = n - p_L - 1$
- We reject the null hypothesis if the p-value is above α , where

$$\text{p-value} = P(F_{df_S - df_L, df_L} > F_{obs})$$

The Wald test

$$H_0: \beta_2 = \beta_3 = \beta_4 = 0$$

Truth
 $\beta_2 = \beta_3 = 0$
 $\beta_4 \neq 0$ ✓

For a vector of coefficients, we can test $H_0: \beta = \beta_0$:

- Use the test statistic

$$W = (\hat{\beta} - \beta_0)^T [\text{Var}(\hat{\beta})]^{-1} (\hat{\beta} - \beta_0)$$

- Under the null, this test statistic has an asymptotic χ_p^2 distribution
- In practice, we replace $\text{Var}(\hat{\beta})$ with $\widehat{\text{Var}}(\hat{\beta})$ and use an F distribution

The LRT

(still global)

If we are using maximum likelihood estimation (we'll cover this soon – turns out to be least squares in MLR), we can use a LRT:

- Use the test statistics

$$\Delta = -2 \log \frac{L_0}{L_1} = -2(l_0 - l_1)$$

- This test statistic has an asymptotic χ_d^2 distribution where d is the difference in the number of parameters between the two models.

Inference: departure from assumptions

$\approx 90-95\%$

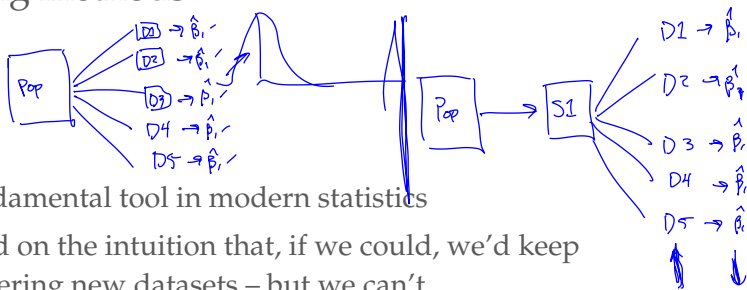


- In large samples, $\hat{\beta}$ is approximately Normal even if the errors are not
- In smaller samples, especially when our assumptions are not justified, the inferential methods we've developed are not valid
- Might also want variance estimates for quantities that are difficult to derive analytically

$$H_0: \underbrace{\beta_2 \cdot \beta_3}_{=0} = 0$$

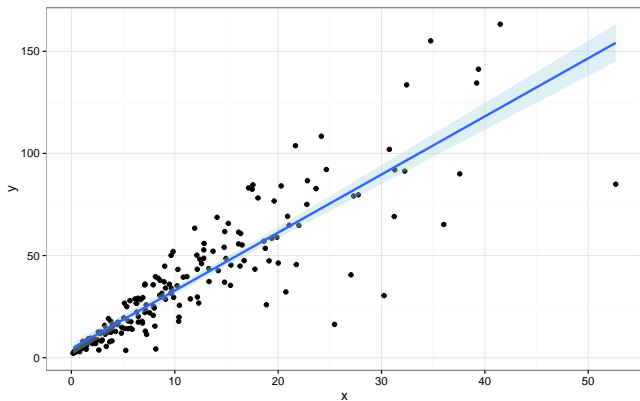
$$H_0: \text{Max}(\beta_n) - \text{min}(\beta_n)$$

Resampling methods

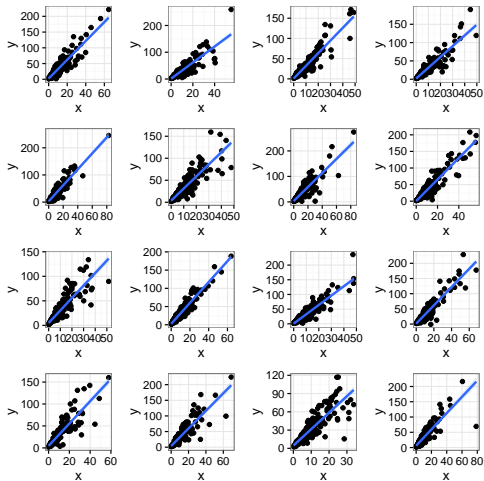


- Fundamental tool in modern statistics
- Build on the intuition that, if we could, we'd keep gathering new datasets – but we can't
- Use repeated samples of a training set to understand variability
- Computationally intensive ... but we have computers

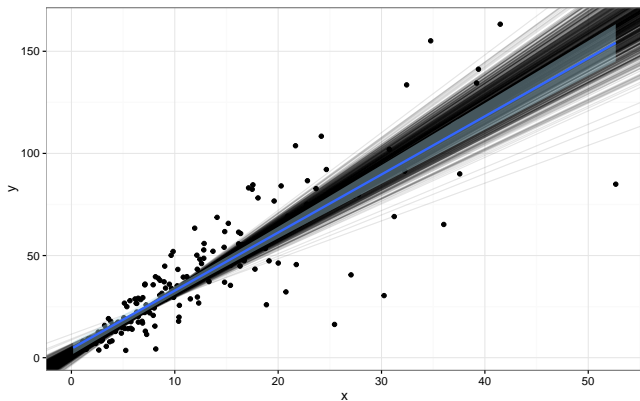
Motivating the Bootstrap



Motivating the Bootstrap



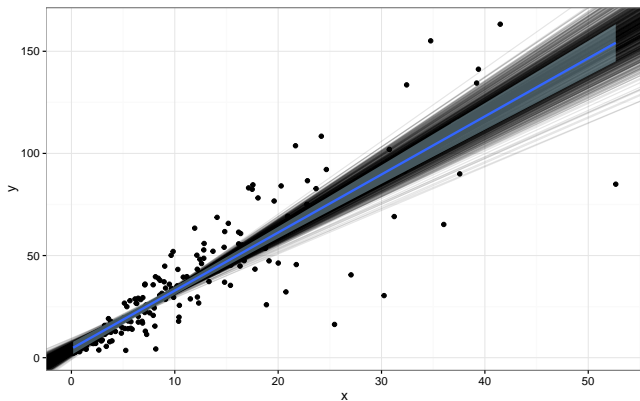
Motivating the Bootstrap



The Bootstrap

- The basic idea is that the observed data mimics the underlying distribution, whatever that may be
- Drawing samples (with replacement) from the observed data mimics drawing samples from the underlying distribution
- Recalculating regression parameters for the “new” samples gives an idea of the distribution of regression coefficients

Implementing the Bootstrap

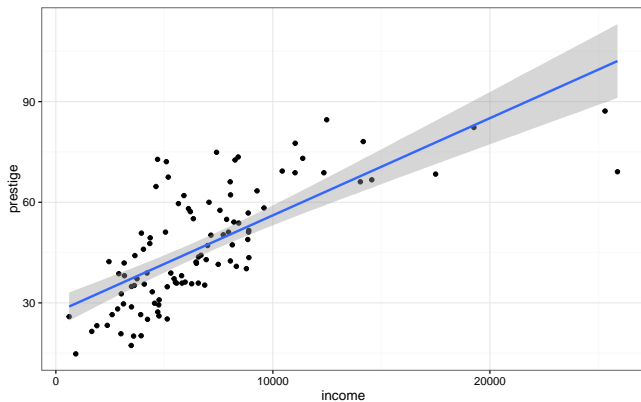


Bootstrap example

Prestige dataset

- Information on 102 occupations
- Variables include education, income, proportion women, job type, and prestige
- Source: 1971 Canadian census

Non-normal inference

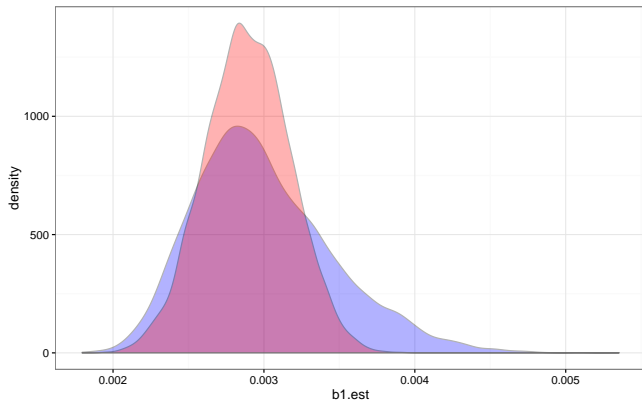


Bootstrap code

```
## define a vector for the bootstrapped estimates
betaHatBS = data.frame(b1.est = rep(NA, 10000))

## use a loop to do the bootstrap
for(i in 1:10000){
  data.cur = sample_frac(Prestige, size = 1, replace = TRUE)
  betaHatBS$b1.est[i] = lm(prestige ~ income, data = data.cur)$coef[2]
}
```

Bootstrap results



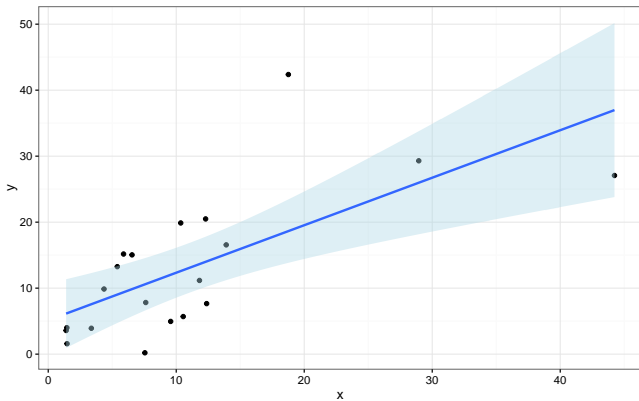
Permutation testing

- Bootstrapping helps understand variability
- What about testing?
- One option – invert the CI from a bootstrap
- Another option – understand distribution of “test statistic” under the null

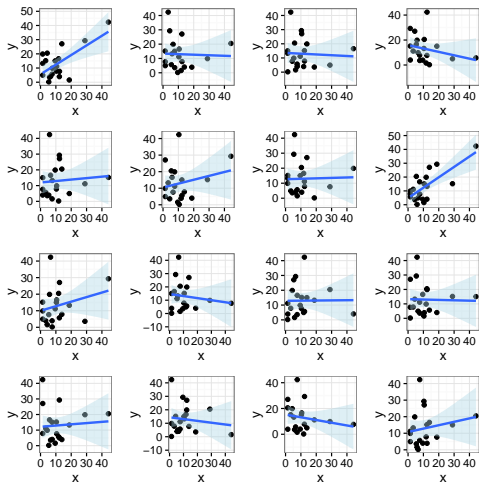
Permute the data

- If we permute the data, there should be no association
- Easy for comparing two groups or SLR; harder for MLRs

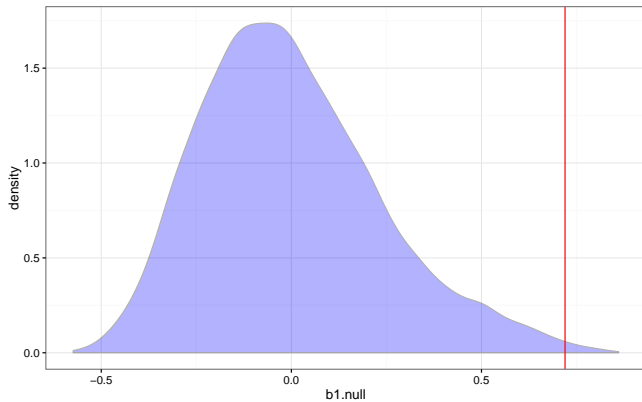
Permutation test example



Permutation test example



Permutation test example



Implementing permutation tests

```
## do enough permutations to test
obs.coef = coef(lm(y ~ x, data = data.noncst))[2]

b1 = data.frame(b1.null = rep(NA, 10000))
for(i in 1:10000){
  data.noncst.cur = mutate(data.noncst, x = sample(x, length(x), replace = FALSE))
  b1$b1.null[i] = coef(lm(y ~ x, data = data.noncst.cur))[2]
}
```

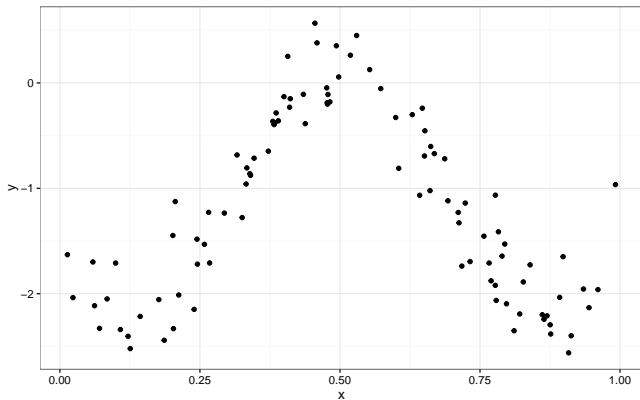

Cross Validation

- Focus is on model performance, quantified by prediction error
- We get in-sample performance ...
- But we want generalization to new data
- Most of the time, we don't have an external testing dataset

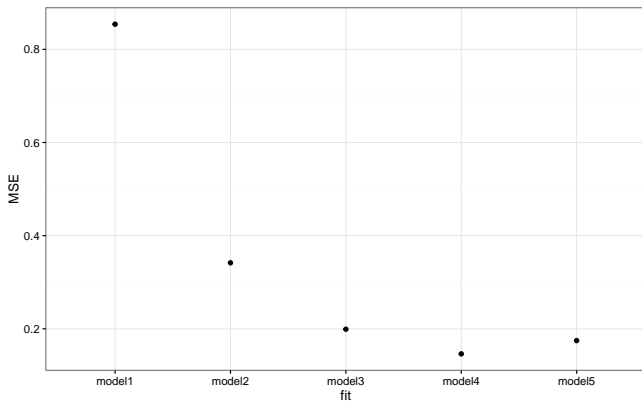
Cross Validation: one validation set

- Simplest case: create a validation set by randomly splitting the full dataset
- Fit model to training data; compute mean squared prediction error on test set
- Provides way of comparing models (any models ...)

Example data

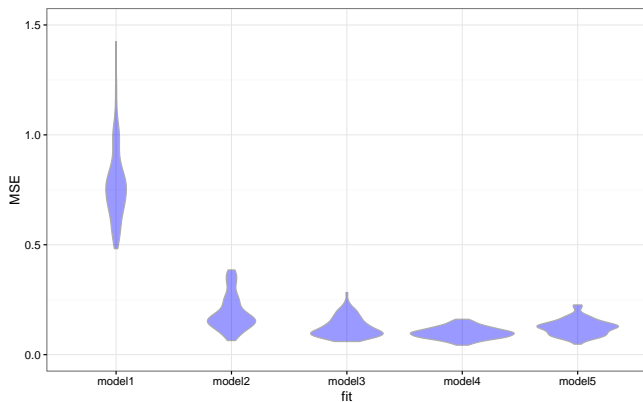


Cross Validation: one validation set



Cross Validation: many validation set

- How you split the data is random; can repeat to understand this source of uncertainty



Implementing CV

```
MSEs = data.frame(  
  modell = rep(NA, 100),  
  ...  
)  
  
for(i in 1:100){  
  
  set.seed(i)  
  data.nonlin = mutate(data.nonlin,  
    cv_group = sample(1:100, 100, replace = FALSE) <= 80,  
    cv_group = factor(cv_group, levels = c(TRUE, FALSE),  
      labels = c("train", "test"))  
  
  data.train = filter(data.nonlin, cv_group == "train")  
  data.test = filter(data.nonlin, cv_group == "test")  
  
  fit.1 = lm(y ~ x, data = data.train)  
  MSEs[i,1] = mean((data.test$y - predict(fit.1, newdata = data.test))^2)  
  
  fit.2 = lm(y ~ x + spline_5, data = data.train)  
  MSEs[i,2] = mean((data.test$y - predict(fit.2, newdata = data.test))^2)  
  
  ...  
}
```

Cross Validation: folds

- Could use k -fold cross validation:
 - ▶ Divide data into k equal-sized folds
 - ▶ Use each one in turn as the validation set; average MSE across sets
 - ▶ k of 5 or 10 is pretty common

Today's big ideas

- Resampling methods
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- Suggested reading: ISLR chapter 5