

# Linear Regression Models

## P8111

Lecture 21

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# Today's Lecture

- LDA review ✓
- Model interpretation
- Model estimation
- Example (pig data!)

# Recall the setting

- We observe data  $y_{ij}^{\uparrow}, x_{ij}$  for subjects  $i = 1, \dots, I$  at visits  $j = 1, \dots, J_i$
- Overall, we pose the model

$$y = X\beta + \epsilon$$

where  $\text{Var}(\epsilon) = \sigma^2 V$  and

$$V = \begin{bmatrix} V_1 & 0 & \dots & 0 \\ 0 & V_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & V_I \end{bmatrix}$$

Uniform = exchangeable

$$\sigma^2 V_i$$
$$\begin{bmatrix} V_i \\ \vdots \\ V_i \end{bmatrix}_{J_i \times J_i}$$

# General ideas

$$y_{ij} | b_i = \beta_0 + \underline{b_i} + \beta_1 x_{ij} + \underline{\epsilon_{ij}}$$

- We discussed two approaches:  $\text{Var}(y_{ij})$ ;  $\text{cov}(y_{ij}, y_{ij'}) \neq 0$ 
  - ▶ Random (mixed) effects models introduce random subject effects and assume uncorrelated errors
  - ▶ Marginal models include only fixed effects and assume correlated subject-level errors

$$y_{i\cdot} = \beta_0 + \beta_1 x_{i\cdot} + \epsilon_{i\cdot}$$

- In some cases these introduce equivalent correlation structures; an example is that a random intercept model induces uniform within-subject correlations
- ✓ Marginal models separate the mean structure and correlation; random effect models induce correlation by introducing random quantities into the mean

# Some comparisons

- Marginal models may be less immediately obvious or interpretable
- However, marginal models can be more robust to misspecification through robust SE estimates

# Marginal model

The marginal model formulation is

$$\underline{\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}}$$

where

$$\underline{\boldsymbol{\epsilon} \sim \mathbf{N}[0, \sigma^2 \mathbf{V}]}$$

$$\left[ \begin{array}{l} E(y_{j_s} | x_{i_j} = 1) - E(y_{j_s} | x_{i_j} = 0) \\ = \beta_t \end{array} \right.$$

- This approach focuses on the *marginal* distribution of  $\mathbf{y}$ , rather than on a subject-level *conditional* distribution
- Coefficients have a marginal interpretation – compare subjects based only on covariate values
- ■ Interpretation is analogous to a cross-sectional model

# Remember GLS

Given the model

$$\underline{y = X\beta + \epsilon}$$

where  $\epsilon \sim N(0, \sigma^2 V)$  with  $V$  known, we are essentially assuming

$$\underline{y \sim N(X\beta, \sigma^2 V)}$$

Using MLE, we find that  $\hat{\beta}_{GLS} = \underline{(X^T V^{-1} X)^{-1} X^T V^{-1}}$

$$L(\beta | y) \propto \exp \dots \dots$$

# Estimation – marginal model

- If we can use MLE when  $V$  is known, maybe we can use MLE to estimate  $V$  as well
- Our log likelihood function is

$$l(\underline{\beta}, \underline{\sigma^2}, \underline{V}; \underline{y}, \underline{X}) = -\frac{1}{2} \left[ n \log(\underline{\sigma^2}) + \log(|\underline{V}|) + \frac{1}{\underline{\sigma^2}} (\underline{y} - \underline{X}\underline{\beta})^T \underline{V}^{-1} (\underline{y} - \underline{X}\underline{\beta}) \right]$$

- Using profile likelihood, we find that for any  $V_0$

$$\hat{\underline{\beta}}(V_0) = (\underline{X}^T V_0^{-1} \underline{X})^{-1} \underline{X}^T V_0^{-1} \underline{y}$$



# Estimation – marginal model

$$L(V, \sigma^2 | y, X)$$

- Estimation of  $V$  and  $\sigma$  is done through restricted maximum likelihood
  - ▶ Standard MLE produces biased variance estimates; REML adjusts for the number of fixed effects components that are estimated
- Often  $V$  is structured parametrically to ease estimation and computation
- We won't worry about how this is done

$$V(\rho)$$

# Random intercept model

$$E(y_{ij} | \boxed{b_i}, x_{ij}) = \beta_0 + \overset{\downarrow}{b_i} + \beta_1 x_{ij}$$

$$E(y_{ij} | \overset{\downarrow}{b_i}, x_{ij}=1) - E(y_{ij} | \overset{\downarrow}{b_i}, x_{ij}=0) = \beta_1$$

A random intercept model with one covariate is given by

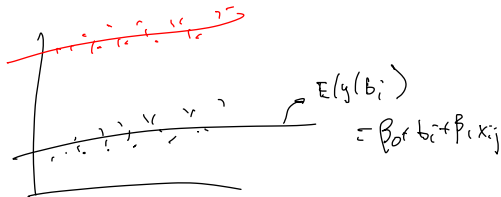
$$\underline{y_{ij}} = \underline{\beta_0} + \underline{b_i} + \underline{\beta_1} x_{ij} + \underline{\epsilon_{ij}}$$

where

- $b_i \sim N[0, \tau^2]$
- $\epsilon_{ij} \sim N[0, \nu^2]$

$$E(y_{ij} | b_i = 0) = \beta_0 + \beta_1 x_{ij}$$

$$\text{Var}(y_{ij} | b_i) = \nu^2 \mathbf{I}$$



# Random intercept model

More compactly, we write

$$\underline{y} = X\underline{\beta} + Z\underline{b} + \underline{\epsilon}$$

$Z$   
 $\sum_{j=1}^S x_j I$

where

- $\underline{b} \sim N[0, \sigma^2 I]$
- $\underline{\epsilon} \sim N[0, \nu^2 I_n]$

$$\begin{array}{c}
 \begin{bmatrix} y_{11} \\ \vdots \\ y_{1S_1} \\ \vdots \\ y_{IS_I} \end{bmatrix} = \begin{bmatrix} x_{111} & \dots & x_{11P} \\ \vdots & & \vdots \\ x_{1S_1 1} & \dots & x_{1S_1 P} \\ \vdots & & \vdots \\ x_{IS_I 1} & \dots & x_{IS_I P} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_P \end{bmatrix} + \begin{bmatrix} \epsilon_{11} \\ \vdots \\ \epsilon_{1S_1} \\ \vdots \\ \epsilon_{IS_I} \end{bmatrix} \\
 \\
 \underline{y} = X\underline{\beta} + Z\underline{b}
 \end{array}$$

# Random intercept model

In the model

$$y = X\beta + \overset{\downarrow}{Zb} + \epsilon$$

why might we assume  $b$  are random rather than estimating them as fixed effects?

Interpretation

'Penalization' / shrinkage / Borrowing Strength

# Parameters

# Random intercept model

In the model

$$y = X\beta + Zb + \epsilon$$

why might we assume  $b$  are fixed rather than considering them to be random?

~~no~~ you don't want penalization

~~no~~ "Random" interp doesn't make sense

(mostly don't do this)

# Random intercept model interpretation

Random effect models have a conditional interpretation.

- Mean conditions on subject effects:  $E(y_{ij}|x_{ij}, \beta, b_i)$
- To derive a marginal (averaged across subjects) mean, one can use iterated expectations
  
- (For an identity link, the coefficients also have a marginal interpretation – this is not true for generalized RE models)

# Estimation – random effect model

Still done using MLE, but now we include random effects; our model is

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{b} + \boldsymbol{\epsilon}$$

where

- $\mathbf{b} \sim \text{N} [0, \tau^2 I_I]$
- $\boldsymbol{\epsilon} \sim \text{N} [0, \nu^2 I_n]$

# Estimation – random effect model



# Estimation – random effect model

## Estimation – BLUPs

- Our estimate for fixed and random effects are

$$\begin{bmatrix} \hat{\beta} \\ \hat{b} \end{bmatrix} = \left( \mathbf{C}^T \mathbf{C} + \frac{\nu^2}{\tau^2} \mathbf{R} \right)^{-1} \mathbf{C}^T \mathbf{y}$$

- These are referred to as “BLUPs” (the “P” is for “predictions”)
- The variances  $\nu^2$  and  $\tau^2$  are estimated via REML

# BLUPs

BLUPs are a lot like BLUEs

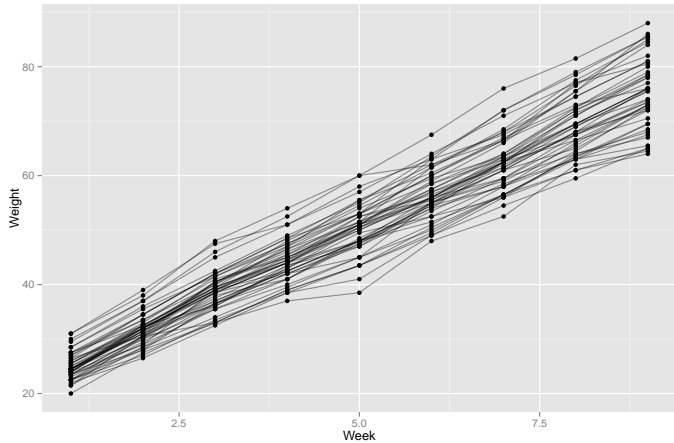
- They are the best linear unbiased predictors when one has both fixed and random effects
- We derived them using Normal distributions, but even without distributional assumptions these are BLUP
- The Normal distributions help with the assumptions are satisfied, in that one can get distribution-based inference

# Penalized spline regression

Note that BLUPs look an awful lot like penalized regression estimates

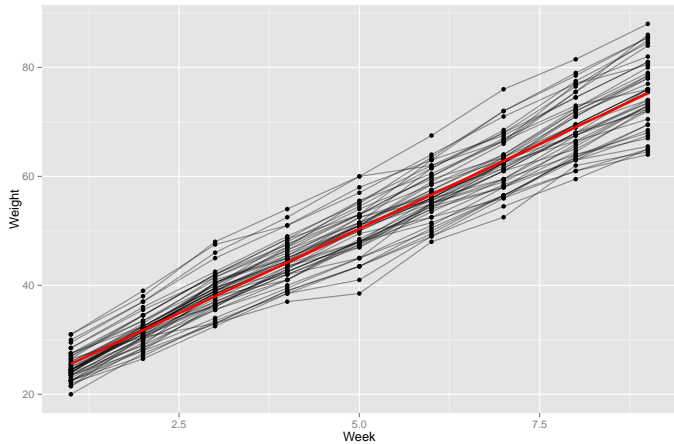
- The “mixed model framework” is commonly used for penalized spline estimation
- The “tuning parameter” is a ratio of variances, and can be estimated via REML (rather than CV)
- This approach provides a method for inference

# Pig data



# Pig data

## OLS fit for pig data



# Pig data

## OLS code

```
> lin.mod = lm(weight ~ num.weeks, data = pig.weights)
> summary(lin.mod)
```

Residuals:

Min	1Q	Median	3Q	Max
-11.9051	-2.5348	-0.1952	2.5949	13.1751

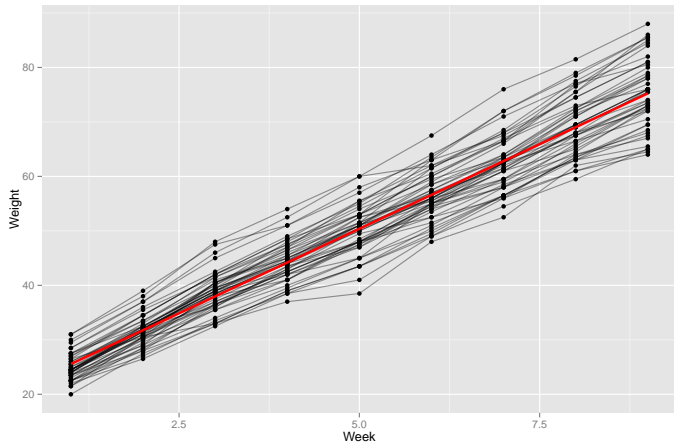
Coefficients:

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	19.35561	0.46054	42.03	<2e-16 ***
num.weeks	6.20990	0.08184	75.88	<2e-16 ***

Residual standard error: 4.392 on 430 degrees of freedom

# Pig data

## Marginal model fit for pig data





# Pig data

## Marginal model code

```
> library(gee)
> marg.mod = gee(weight ~ num.weeks, id = id.num, corstr = "exchangeable",
  data = pig.weights)
> summary(marg.mod)
```

Model:

```
Link: Identity
Variance to Mean Relation: Gaussian
Correlation Structure: Exchangeable
```

Summary of Residuals:

	Min	1Q	Median	3Q	Max
	-11.9050926	-2.5347801	-0.1951968	2.5949074	13.1751157

Coefficients:

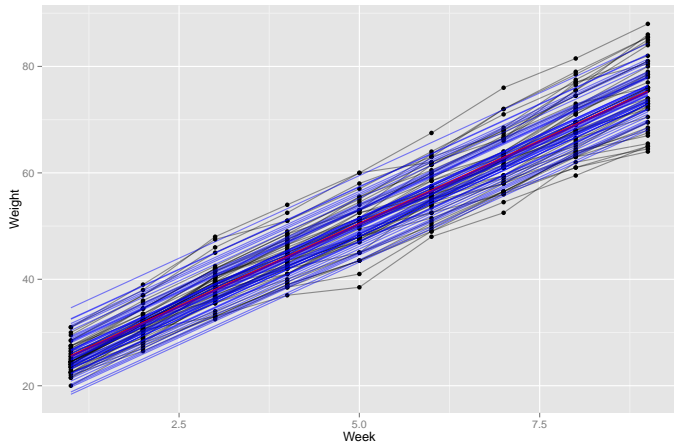
	Estimate	Naive S.E.	Naive z	Robust S.E.	Robust z
(Intercept)	19.355613	0.5983680	32.34734	0.39963854	48.43280
num.weeks	6.209896	0.0393321	157.88366	0.09107443	68.18485

Working Correlation

	[,1]	[,2]	[,3]	[,4]	...
[1,]	1.0000000	0.7690313	0.7690313	0.7690313	...
[2,]	0.7690313	1.0000000	0.7690313	0.7690313	...
[3,]	0.7690313	0.7690313	1.0000000	0.7690313	...
[4,]	0.7690313	0.7690313	0.7690313	1.0000000	...
...					

# Pig data

Mixed effect model fit for pig data



# Pig data

## Mixed effect model code

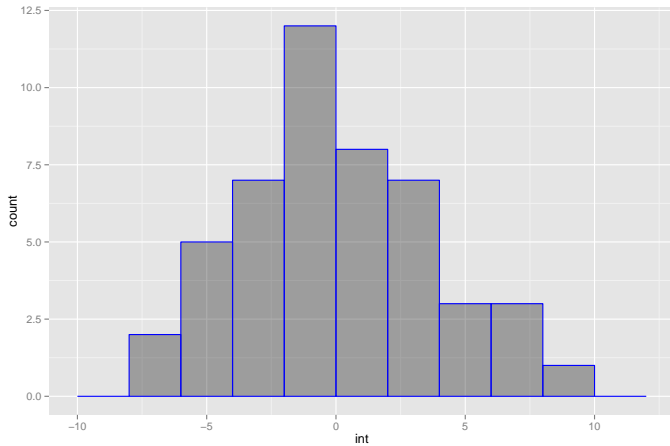
```
> library(lme4)
> ranef.mod = lmer(weight ~ (1 | id.num) + num.weeks, data = pig.weights)
> summary(ranef.mod)
Linear mixed model fit by REML
Formula: weight ~ (1 | id.num) + num.weeks
Data: pig.weights
AIC BIC logLik deviance REMLdev
2042 2058 -1017 2030 2034
Random effects:
Groups Name Variance Std.Dev.
id.num (Intercept) 15.1418 3.8913
Residual 4.3947 2.0964
Number of obs: 432, groups: id.num, 48

Fixed effects:
Estimate Std. Error t value
(Intercept) 19.35561 0.60311 32.09
num.weeks 6.20990 0.03906 158.97

> (15.1418) / (15.1418 + 4.3947)
[1] 0.7750518
```

# Pig data

## Histogram of estimated random intercepts



# Today's big ideas

- Estimation in LDA
- Example + code

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Potential reading on mixed effects models – Semiparametric Regression (Ruppert, Wand, Carroll) Ch 4