

# Linear Regression Models

## P8111

- HW 4 notes
  - no skewed CV
  - outlier removal
  -

### Lecture 23

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# Today's Lecture

- Multilevel models ✓
  - ▶ Hierarchical / nested models
  - ▶ Crossed designs ✓
- Bayesian methods

# Longitudinal data

- We observe data  $y_{ij}, x_{ij}$  for subjects  $i = 1, \dots, I$  at visits  $j = 1, \dots, J_i$
- Overall, we pose the model

$$\underline{y = X\beta + \epsilon}$$

where  $\underline{\text{Var}(\epsilon) = \sigma^2 V}$  and

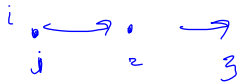
$v(\rho)$

$$V = \begin{bmatrix} V_1 & 0 & \dots & 0 \\ 0 & V_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & V_I \end{bmatrix}$$

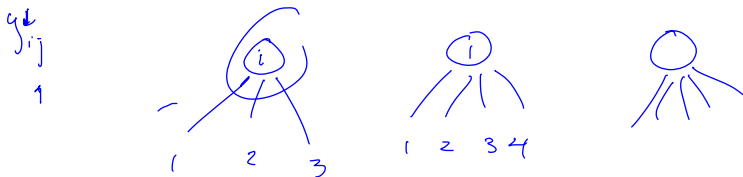
# Longitudinal data

- Extended cross-sectional models to allow repeated subject observations
- Repeated observations had a time element
- One basic approach was random effects

# Multilevel models



- Multilevel models are a 'more general' class of models
- Repeated observations don't necessarily have to be taken in time
- Examples of two-level models include students in a class, members in a family, patients in a hospital, etc



# Two-level model

The repeated observations structure we developed for longitudinal data helps for two-level models.

Specifically for repeated observations  $j$  within clusters  $i$ , we could write

$$y_{ij} = \beta_0 + \beta_1 x_{ij} + b_i + \epsilon_{ij}$$

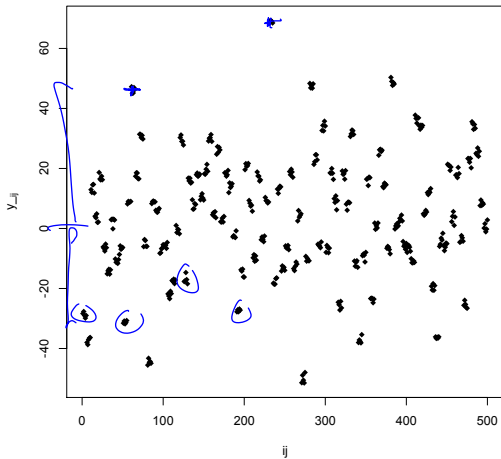
with

- $b_i \sim N[0, \tau^2]$
- $\epsilon \sim N[0, \nu^2]$

Intuition, estimation, induced correlation, interpretation – all of these were established for LDA and transfer here

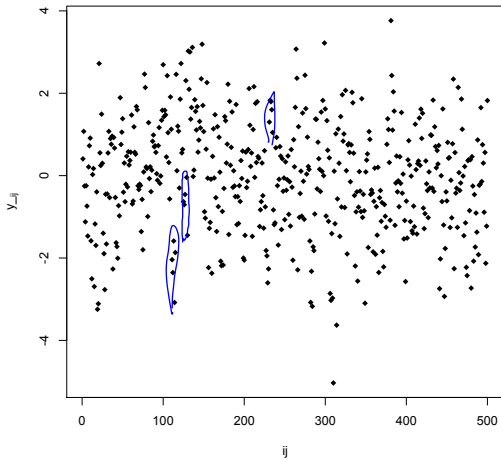
# Example I

$$\tau^2 \Rightarrow \sigma^2 \quad / \quad ICC \approx 1$$



# Example II

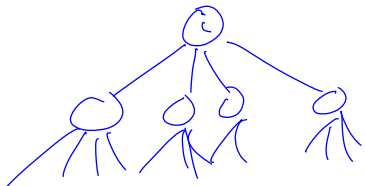
$$\gamma^2 \ll \nu^2 \quad / \quad \text{ICC} \approx 0$$





# Three level model

- Sometimes, the data have a more complex nested structure
- Each cluster is part of a larger cluster
- Examples include students in classes in universities, members in families in towns, patients in hospitals in regions





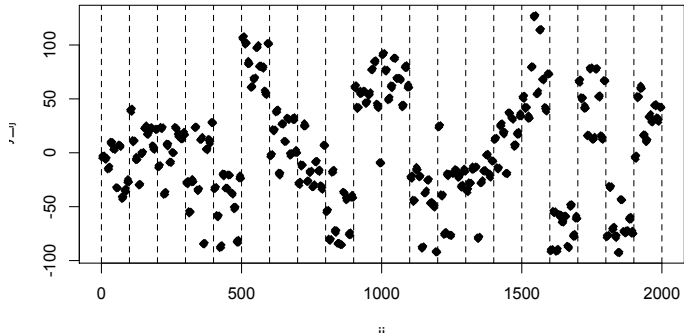
# ICCs

This model gives two levels of correlation (observations within clusters, clusters within super clusters), and therefore a couple of ICCs:

- $cov(y_{ijk}, y_{ijk'}) =$

- $cov(y_{ijk}, y_{ij'k}) =$

# Example



# Example

## Nested model

```
> nested.mod = lmer(yij ~ (1 | L1) + (1 | L2))
> summary(nested.mod)
Linear mixed model fit by REML ['lmerMod']
Formula: yij ~ (1 | L1) + (1 | L2)
```

REML criterion at convergence: 7464.337

Random effects:

Groups	Name	Variance	Std.Dev.
L2	(Intercept)	527.003	22.957
L1	(Intercept)	2137.453	46.233
	Residual	1.004	1.002

Number of obs: 2000, groups: L2, 200; L1, 20

Fixed effects:

	Estimate	Std. Error	t value
(Intercept)	0.7712	10.4646	0.074

# Crossed designs

- Alternatively to nested (hierarchical) models, sometimes there is a crossed design
- Each subject is observed under multiple “treatments”, so there are both subject and treatment effects
- For example, each student is graded in multiple classes; each patient is assayed for multiple genes

# Crossed designs

For a crossed model (with subjects  $i$  and treatments  $j$ ), we can write

$$y_{ij} = \beta_0 + \beta_1 x_{ij} + b_i + b_j + \epsilon_{ij}$$

with

- $b_i \sim \text{N} \left[ 0, \tau_{(1)}^2 \right]$
- $b_j \sim \text{N} \left[ 0, \tau_{(2)}^2 \right]$
- $\epsilon \sim \text{N} \left[ 0, \nu^2 \right]$

Here there is covariance within subjects across treatments, and within treatments across subjects.

# ICCs

Here there is covariance within subjects across treatments, and within treatments across subjects.

- $cov(y_{ij}, y_{ij'}) =$

- $cov(y_{i'j}, y_{ij}) =$



# LDA and MLM

- Estimation works basically the same for these models as for random intercept models
- Intuition is the same as well – you want to borrow strength for one subject from the population of other subjects
- Interpretation of fixed effects is *marginal*; interpretation of random effects is *conditional*
- Using randomness both decreases the number of parameters and induces correlation structures

# Bayesian methods

Longitudinal data analysis and multilevel models are a good place to start “thinking Bayesian”

- Even though they’re frequentist, they include randomness at subject levels
- The idea of “shrinking toward a population mean” or “borrowing strength” is a pretty Bayesian concept
- Even writing down random effect distributions is reminiscent of defining prior distributions

# Basic Bayes

LDA and MLM are fairly advanced topics, so we'll start with a simpler example

- Suppose I gather data  $y_i$  and want to learn about  $E(y)$
- Suppose even more I think I already know *something* about  $E(y)$
- I might write down something about what I want to learn and what I think I know

# Basic Bayes

What do I think I know?

- $y_i | \mu \sim \text{N} [\mu, \sigma_y^2]$
- $\mu \sim \text{N} [\mu_0, \sigma_0^2]$

What do I want to learn?

- $\mu | y_i \sim ???$

# Basic Bayes

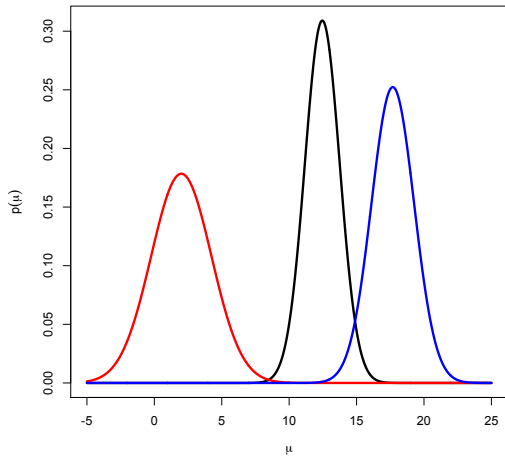
Luckily, this is all related through Bayes' formula:

$$p(\mu|y_i) \propto p(y_i|\mu)p(\mu)$$

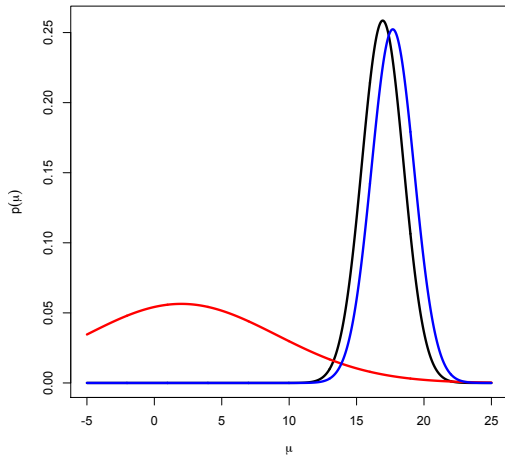
- For the Normal likelihood with a Normal prior for the mean, the posterior is also Normal:

$$\mu|y_i \sim \text{N} \left[ \frac{\sigma_\mu^2}{\frac{\sigma_y^2}{n} + \sigma_\mu^2} \bar{y} + \frac{\frac{\sigma_y^2}{n}}{\frac{\sigma_y^2}{n} + \sigma_\mu^2} \mu_0, \frac{\frac{\sigma_y^2}{n} \sigma_\mu^2}{\frac{\sigma_y^2}{n} + \sigma_\mu^2} \right]$$

# Effect of Prior



# Effect of Prior



# Bayesian regression

How can we pose the regression model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

with  $\boldsymbol{\epsilon} \sim \text{N}[0, I_n]$  in a Bayesian framework?

- By making distributional assumptions about the  $\boldsymbol{\beta}$
- Normal priors seemed to work well in the past ...
- Try  $\boldsymbol{\beta} \sim \text{N}\left[0, \sigma_{\boldsymbol{\beta}}^2 I_p\right]$  where  $p$  includes the intercept



# Bayesian regression

We want to obtain the posterior

$$p(\boldsymbol{\beta}|\mathbf{y}, \mathbf{X}) \propto p(\mathbf{y}|\boldsymbol{\beta}, \mathbf{X})p(\boldsymbol{\beta})$$

# Bayesian regression

Can show that

$$[\boldsymbol{\beta} | \mathbf{y}, \mathbf{X}] \sim \mathbf{N} [\boldsymbol{\mu}_p, \boldsymbol{\Sigma}_p]$$

where

$$\boldsymbol{\Sigma}_p = \left( \frac{1}{\sigma_\epsilon^2} \mathbf{X}^T \mathbf{X} + \frac{1}{\sigma_\beta^2} \mathbf{I} \right)^{-1}$$

and

$$\boldsymbol{\mu}_p = \boldsymbol{\Sigma}_p \left( \frac{1}{\sigma_\epsilon^2} \mathbf{X}^t \mathbf{y} \right)$$

## So, about the variances

- Throughout all of this we have implicitly conditioned on the variances  $\sigma_\epsilon^2$  and  $\sigma_\beta^2$
- Doesn't affect any of our calculations – the terms involving  $\mu$  don't overlap with terms involving  $\sigma_\epsilon^2$  or  $\sigma_\beta^2$
- $\sigma_\beta^2$  is often treated as fixed;  $\sigma_\epsilon^2$

# The full posterior

- Need  $[\beta, \sigma_\epsilon^2 | \mathbf{y}, \mathbf{X}]$
- “Intractable” problem
- Just as good: sample from the posterior

# Sampling from the posterior

- aka where Bayes gets really weird
- You can draw a sample from the posterior even if you can't write down exactly what it is
- That sample is your basis for inference
  - ▶ Posterior sample average is your estimate
  - ▶ Quantiles on the posterior sample define your credible interval
- Sample describes the *joint distribution* of all model parameters

## Some notes on this business

- Joint distributions are often worth the trouble
- Bayesian methods were really controversial for a long time, but are at least less controversial now
- The introduction of “prior knowledge” happens even in frequentist methods, although it is often not explicitly acknowledged

# Today's big ideas

- Nested and crossed random effects models
  - Bayesian stuff
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